

NONLINEAR CLASSIFICATION OF BANACH SPACES

A Dissertation

by

NIRINA LOVASOA RANDRIANARIVONY

Submitted to the Office of Graduate Studies of
Texas A&M University
in partial fulfillment of the requirements for the degree of

DOCTOR OF PHILOSOPHY

August 2005

Major Subject: Mathematics

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ABSTRACT

Nonlinear Classification of Banach Spaces. (August 2005)

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Chair of Advisory Committee: Dr. William B. Johnson

We study the geometric classification of Banach spaces via Lipschitz, uniformly continuous, and coarse mappings. We prove that a Banach space which is uniformly homeomorphic to a linear quotient of ℓ_p is itself a linear quotient of ℓ_p when $p < 2$. We show that a Banach space which is Lipschitz universal for all separable metric spaces cannot be asymptotically uniformly convex. Next we consider coarse embedding maps as defined by Gromov, and show that ℓ_p cannot coarsely embed into a Hilbert space when $p > 2$. We then build upon the method of this proof to show that a quasi-Banach space coarsely embeds into a Hilbert space if and only if it is isomorphic to a subspace of $L_0(\mu)$ for some probability space $(\Omega, \mathcal{B}, \mu)$.

For my family

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CHAPTER I

INTRODUCTION

How rigid is the linear structure of a Banach space? That is the question we set to investigate in this study. Banach spaces are considered as metric spaces, and considered between them are maps which control the distances between points in a uniform manner. The question is to study which aspects of the linear geometry of these spaces have to be transported by such maps from one of the Banach spaces into the other.

Such questions have been widely studied. We mention in particular the Mazur-Ulam Theorem [MaU] which says that if the map considered is a surjective isometry, then *all* of the aspects of the linear geometry of one of the Banach spaces are transported into the other, namely the two Banach spaces are linearly isometric.

In the present study, the control we impose on a map $f : X \rightarrow Y$ between two Banach spaces X and Y is not quite as strict as an isometry. We relax a bit by allowing (uniform) lower and upper bounds on the distances. Namely, we request that for any $x, y \in X$ we have:

$$\varphi_1(\|x - y\|) \leq \|f(x) - f(y)\| \leq \varphi_2(\|x - y\|),$$

where $\varphi_1, \varphi_2 : [0, \infty) \rightarrow [0, \infty)$ are nondecreasing functions.

The study divides into two parts. In Chapters II and III, we focus our attention to controlling small distances. Namely we request that $\varphi_1(t) > 0$ when $t > 0$, and

^{*}The journal model is Geometric and Functional Analysis.

that $\varphi_2(t) \rightarrow 0$ as $t \rightarrow 0$. The maps that have such properties are exactly the uniform and/or Lipschitz embeddings.

In Chapter II, we study Banach spaces that are uniformly homeomorphic to a linear quotient of ℓ_p when $1 < p < 2$. Such spaces are of course uniform quotients of ℓ_p as well (see 2.2.3 for the formal definition). The case $p > 2$ has been studied. Namely, in [GoKL2], Godefroy, Kalton and Lancien show that if a Banach space X is uniformly homeomorphic to a quotient of ℓ_p when $p > 2$, then X has to be a linear quotient of ℓ_p as well. Johnson, Lindenstrauss, Preiss and Schechtman in [JoLPS] show that ℓ_2 cannot be a Lipschitz quotient of ℓ_p when $p > 2$. The techniques for these were respectively the conservation of the convex Szlenk index under a uniform homeomorphism, and the ϵ -Fréchet differentiability of a Lipschitz map from ℓ_p to ℓ_2 when $p > 2$. In our study of the case $p < 2$, we employ the technique introduced by Enflo in his proof that ℓ_1 and L_1 are not uniformly homeomorphic, namely we use a comparison of the sizes of approximate metric midpoints. Using this technique, we get that a Banach space which is uniformly homeomorphic to a linear quotient of ℓ_p must itself be a linear quotient of ℓ_p when $1 < p < 2$.

In Chapter III, we refine the Enflo technique by relating the sizes of approximate metric midpoints with the moduli introduced by Milman in [Mi]: the modulus of asymptotic uniform convexity $\bar{\delta}_X$ and the modulus of asymptotic uniform smoothness $\bar{\rho}_X$ of a Banach space X . Through this refinement, the Enflo technique gives us an estimate of the form $\bar{\delta}_Y(t) \leq C\bar{\rho}_X(Ct)$ when the Banach space X Lipschitz embeds into the Banach space Y . This enables us to give one characteristic of Banach spaces that contain a Lipschitz copy of every separable metric space.

In Chapters IV and V, we change focus and control large distances instead. Now our requirement on the control that the map $f : X \rightarrow Y$ makes on distances

is that $\varphi_1(t) \rightarrow \infty$ as $t \rightarrow \infty$. These maps were introduced by Gromov [Gr1] and are called coarse embeddings. They were introduced in order to study groups as geometric objects. Finitely generated groups are considered as metric spaces under the word distance. In relation to algebraic topology, Yu [Y] proved that a metric space with bounded geometry that coarsely embeds into a Hilbert space satisfies the coarse geometric Novikov conjecture. Later, Kasparov and Yu [KaY] strengthened this result by showing that it is enough for the metric space in question to admit a coarse embedding into a uniformly convex Banach space. Whether this was really a strengthening was not very clear though, because it is not apparent at first sight that there are uniformly convex Banach spaces that do not coarsely embed into a Hilbert space.

This inspires the study we do in Chapters IV and V. In Chapter IV, we show that the uniformly convex Banach space ℓ_p ($p > 2$) does not admit a coarse embedding into a Hilbert space. In Chapter V, we give an actual characterization of all quasi-Banach spaces that coarsely embed into a Hilbert space. The techniques used here rely on the classical work of Schoenberg [S] regarding the theory of positive and negative definite kernels; and on the work of Aharoni, Maurey and Mityagin [AMM] regarding the study of uniform embeddings into a Hilbert space.

CHAPTER II

BANACH SPACES UNIFORMLY HOMEOMORPHIC TO A QUOTIENT OF ℓ_p ($1 < p < 2$)

Let us recall the definition and the main property of a uniform homeomorphism.

2.1 Uniform homeomorphism

Definition 2.1.1. A map f is called a *uniform homeomorphism* between two metric spaces X and Y if it is bijective, uniformly continuous, and its inverse is also uniformly continuous. If f and f^{-1} are actually Lipschitz maps, then we say f is a *Lipschitz homeomorphism* or a *Lipschitz isomorphism*.

A map Ω_f called the *modulus of uniform continuity of f* gives a way to quantify the uniform continuity of f . It is defined for $r \in [0, \infty)$ by

$$\Omega_f(r) = \sup\{d(f(x), f(y)), d(x, y) \leq r\}.$$

The map f is uniformly continuous if $\Omega_f(r) \rightarrow 0$ when $r \rightarrow 0$. When the function is Lipschitz, its modulus of uniform continuity has a linear form, i.e. there is a constant $C > 0$ so that $\Omega_f(r) \leq Cr$ for all $r \geq 0$.

The Lipschitz behavior of a map is closer to a linear behavior than the uniform behavior is, so it is natural to “Lipschitz-ize” a uniformly continuous map. The following lemma (see e.g. [BeL, page 18]), which does this in an explicit manner, will be of importance to us throughout this work:

Lemma 2.1.2 (Lipschitz for large distances property). *Let $f : X \rightarrow Y$ be a uniformly continuous map. If the metric space X is metrically convex, then f is*

Lipschitz for large distances, i.e.

$$\forall c > 0, \exists L < \infty \text{ such that } d_X(x, y) \geq c \Rightarrow d_Y(f(x), f(y)) \leq Ld_X(x, y).$$

Proof: Recall that a metric space X is called *metrically convex* if for all x, y in X , and for all $n \in \mathbb{N}$, one can find in X points $x_0 = x, x_1, x_2, \dots, x_n = y$ such that $d(x_i, x_{i+1}) \leq d(x, y)/n$.

Call Ω_f the modulus of uniform continuity of f , and let $c > 0$. Let $x, y \in X$ with $d(x, y) \geq c$, and call n the smallest integer bigger than or equal to $d(x, y)/c$. Since X is metrically convex, we can find in X points $x_0 = x, x_1, x_2, \dots, x_n = y$ such that $d(x_i, x_{i+1}) \leq c$ for all $i = 0, 1, 2, \dots, n-1$. Then by the triangle inequality in Y , we have:

$$\begin{aligned} d(f(x), f(y)) &\leq \sum_{i=0}^{n-1} d(f(x_i), f(x_{i+1})) \\ &\leq \sum_{k=0}^{n-1} \Omega_f(c) \\ &= \Omega_f(c)n \\ &\leq \Omega_f(c) \frac{2d(x, y)}{c} \end{aligned}$$

So we can take $L = 2\Omega_f(c)/c$. □

Remark 2.1.3. A Banach space is convex, and hence metrically convex, so it follows from Lemma 2.1.2 that a uniform homeomorphism f from a Banach space X onto a subset of another Banach space Y is Lipschitz for large distances. However, $f(X)$ need not be a convex subset of Y , and hence we cannot always assume that f^{-1} is Lipschitz for large distances as well. Unless imposed by assumption, the only case where f^{-1} is guaranteed to be Lipschitz for large distances is when f is surjective, i.e. the two Banach spaces are uniformly homeomorphic.

2.2 Quotient mappings

Definition 2.2.1. A Banach space Y is called a *linear quotient* of a Banach space X if there is a linear bounded map $T : X \rightarrow Y$ that is surjective.

Recall the Open Mapping Theorem, which gives a very distinctive property of linear quotient mappings.

Theorem 2.2.2 (Open Mapping Theorem). *Let X and Y be Banach spaces, and let $T : X \rightarrow Y$ be a linear bounded map. If T is surjective, then there exists a constant $c \in (0, \infty)$ such that:*

$$\frac{1}{c}B_Y \subset T(B_X),$$

(where B_X and B_Y represent the unit ball of X and Y respectively).

In other words, linear quotient mappings are uniformly open in the sense that the image of any ball of a given radius centered at a point x contains a ball of proportional radius centered at $T(x)$. This observation leads to the following definition for general nonlinear mappings.

Definition 2.2.3. Let X and Y be Banach spaces. A map $f : X \rightarrow Y$ is called a *uniform quotient map* ([BJLPS], [J]) if it is uniformly continuous, surjective, and uniformly open in the sense that one can find a nondecreasing map $\omega : [0, \infty) \rightarrow [0, \infty)$ such that $\omega(r) > 0$ when $r > 0$, and

$$\forall x \in X, B(f(x), \omega(r)) \subset f(B(x, r)).$$

ω is called the *modulus of co-uniformity* of f .

If f is Lipschitz, and ω is such that $\omega(r) \geq \frac{r}{c}$ for some constant c independent of x and r , then f is called a *Lipschitz quotient map*, and the smallest constant c

which makes the inequality true for all $r \in [0, \infty)$ is called its *co-Lipschitz constant* [Gr2].

Let us remark that if X is a separable Banach space, and Y is a uniform quotient of X , then Y is separable. In fact, all we need for this is the continuity and the surjectivity of the uniform quotient map.

A uniform quotient f map between Banach spaces is also *co-Lipschitz for large distances* [BJLPS]. Roughly speaking, it means that if the map were invertible, its inverse would have been Lipschitz for large distances. Formally, it means the following:

Lemma 2.2.4 (Co-Lipschitz for large distances property). *Let f be a uniform quotient map between two Banach spaces X and Y . Then f is co-Lipschitz for large distances, i.e.*

$$\forall c > 0, \exists L \in (0, \infty) \text{ such that } r \geq c \Rightarrow \forall x \in X, B\left(f(x), \frac{r}{L}\right) \subset f(B(x, r)).$$

Proof: Let $f : X \rightarrow Y$ be a uniform quotient, with modulus of co-uniformity ω . Let $r \geq c$, let $x \in X$, and let $y \in B(f(x), \omega(r))$. Divide the segment $[f(x), y]$ into n segments $y_0 = f(x), y_1, \dots, y_n = y$ with $\|y_i - y_{i-1}\| \leq \omega(c)$ and $n = \left\lceil \frac{\omega(r)}{\omega(c)} \right\rceil$. Now, by the co-uniform condition on f , we can find points $x_0 = x, x_1, \dots, x_n \in X$ such that $f(x_i) = y_i$ and $\|x_i - x_{i-1}\| \leq c$ for every $i = 1, \dots, n$. Using the triangle inequality in X , we then see that $\|x - x_n\| \leq nc \leq 2\frac{c}{\omega(c)}\omega(r)$ and $f(x_n) = y$. Denoting $L = 2\frac{c}{\omega(c)}$, we see that for every $x \in X$ and for every $r \geq c$,

$$B(f(x), \omega(r)) \subset f(B(x, L\omega(r))).$$

Now if we write $R = L\omega(r)$, we see that $R > 2c$ implies that $r > c$ and hence $B\left(f(x), \frac{R}{L}\right) \subset f(B(x, R))$. □

Now we can make use of an ultrapower argument to get the following (see [BJLPS]):

Proposition 2.2.5. *Let X and Y be Banach spaces, and assume Y is a uniform quotient of X . Let \mathcal{U} be a free ultrafilter on \mathbb{N} . Then the ultrapower $(Y)_{\mathcal{U}}$ is a Lipschitz quotient of the ultrapower $(X)_{\mathcal{U}}$.*

Proof (see e.g. [BeL, page 273]): Let $f : X \rightarrow Y$ be the uniform quotient map. We know that f is Lipschitz and co-Lipschitz for large distances, so we can find a constant $L \in (0, \infty)$ such that:

$$B\left(f(x), \frac{r}{L}\right) \subset f(B(x, r)) \subset B(f(x), Lr)$$

for every $x \in X$ and every $r \geq 1$. Then the maps $f_n(x) = \frac{f(nx)}{n}$ also satisfy the same condition for every $r \geq \frac{1}{n}$. By taking limits along the ultrafilter \mathcal{U} , we see that their ultraproduct $\tilde{f} : (X)_{\mathcal{U}} \rightarrow (Y)_{\mathcal{U}}$ defined by $\tilde{f}(x_1, x_2, \dots) = (f_1(x_1), f_2(x_2), \dots)$ satisfies it for every $r > 0$. \square

Recall that a Banach space X is called uniformly convex if for every $\epsilon > 0$, one can find $\delta > 0$ such that if x and y are two elements of the unit sphere of X with $\|x - y\| \geq \epsilon$, then $\left\|\frac{x + y}{2}\right\| \leq 1 - \delta$. Bates, Johnson, Lindenstrauss, Preiss and Schechtman [BJLPS] improved the previous proposition in the case of uniform quotients of a uniformly convex Banach space (see also [BeL, Theorem 11.18]):

Theorem 2.2.6. *Assume Y is a Banach space that is a uniform quotient of a uniformly convex Banach space X . Then there is a constant $c > 0$ such that for each finite dimensional subspace $F \subset Y^*$, there is a subspace $F_1 \subset X^*$ such that $d(F, F_1) \leq c$. In particular, Y is isomorphic to a linear quotient of some ultrapower of X . (Here $d(\cdot, \cdot)$ represents the Banach-Mazur distance between Banach spaces.)*

Idea of the proof: First, the previous proposition gives us a Lipschitz quotient map between ultrapowers. Now, since a Banach space and its ultrapowers have the same finite-dimensional subspaces, we can just reduce to f being a Lipschitz quotient map between X and Y .

Bates, Johnson, Lindenstrauss, Preiss and Schechtman introduce and study a property of pairs of Banach spaces that they call *UAAP* (Uniform Approximation by Affine Property). A pair (X, Y) is said to have the UAAP if for every $\epsilon > 0$ there is a constant $c > 0$ such that for every ball $B \subset X$ with radius r , and for every Lipschitz map $f : B \rightarrow Y$ with Lipschitz constant L , one can find a ball $B_1 \subset B$ (not necessarily cocentric to B) with radius $r_1 > cr$, and a continuous affine map $g : X \rightarrow Y$ such that $\|f(x) - g(x)\| \leq \epsilon r_1 L$ for every $x \in B_1$.

Then they show that if X is a uniformly convex Banach space, and G is a finite-dimensional Banach space, then (X, G) has the UAAP.

Now, for each finite dimensional subspace F of Y^* , they consider the finite dimensional space $G = Y/F_\perp$ where F_\perp is the annihilator of F in Y . They apply the UAAP property of (X, G) to the composition map $Q \circ f$ where $f : X \rightarrow Y$ is the given Lipschitz quotient map and $Q : Y \rightarrow G$ is the canonical quotient map. The affine approximation of $Q \circ f$ that they get gives a linear map $T : X \rightarrow G$, and then the Lipschitz and co-Lipschitz conditions on $Q \circ f$ dualize and give that the adjoint map $T^* : F \rightarrow X^*$ is an isomorphism into. \square

As a corollary to this theorem, we have the following [BJLPS]:

Corollary 2.2.7 (Uniform quotients of ℓ_p). *Let X be a Banach space that is a uniform quotient of ℓ_p ($1 < p < \infty$). Then X is a linear quotient of $L_p[0, 1]$.*

Proof: Recall that ℓ_p is uniformly convex when $1 < p < \infty$, hence we can apply

Theorem 2.2.6 to get that X is a linear quotient of an ultrapower of ℓ_p . But such an ultrapower is isometric to $L_p(\mu)$ for some measure μ [H]. Since X is separable, X is then a linear quotient of $L_p[0, 1]$. \square

To summarize what we have learned so far, some problems in the classification of uniform quotients of a Banach space can be transformed into linear problems. Since the linear classification of Banach spaces has been very extensively studied, this technique then enables us to use the wealth that the linear theory gives us. Some results that are of importance to us will be presented in the following section.

2.3 Results from the linear theory

The following is a result of Johnson and Odell [JoO]:

Theorem 2.3.1 (Subspaces of L_q that embed into ℓ_q). *Let $2 < q < \infty$, and let X be isomorphic to a subspace of L_q . If ℓ_2 is not isomorphic to a subspace of X , then X embeds into ℓ_q .*

Another result that we will use is the Maurey Extension Theorem [M]:

Theorem 2.3.2 (Maurey Extension Theorem). *Let X be a Banach space with type 2, and let Y be a subspace of X . Let H be a Hilbert space. Then for every bounded linear operator $T : Y \rightarrow H$, there exists an operator $S : X \rightarrow H$ which extends T and which satisfies $\|S\| \leq C\|T\|$, where C is the type-2 constant of X . In particular, every subspace of X which is isomorphic to a Hilbert space is complemented (i.e. is the range of a bounded linear projection).*

The last result that we will use in this chapter is the following theorem of Johnson and Zippin [JoZ]:

Theorem 2.3.3. *If Y is a linear quotient of ℓ_p , then Y embeds into an ℓ_p -sum of finite-dimensional spaces.*

2.4 Approximate metric midpoints

In this section, we are going to present an aspect of the geometry of Banach spaces considered as metric spaces.

Definition 2.4.1. Given two points x and y in a Banach space X , and given $\delta \geq 0$, we define the *metric midpoint of x and y with error δ* by

$$\text{Mid}(x, y, \delta) = \left\{ z \in X, \max(\|x - z\|, \|y - z\|) \leq (1 + \delta) \left\| \frac{x - y}{2} \right\| \right\}.$$

The following proposition gives a way to quantify the “sizes” of approximate midpoints in ℓ_p -type spaces ([BeL, Proposition 10.10]):

Proposition 2.4.2. *Let $1 \leq p < \infty$, and let $X = (\sum_k F_k)_p$ be an ℓ_p -sum of the finite dimensional spaces $(F_k)_k$. Then for every $x \in X$ and every $0 < \delta < 1$,*

1. *there is a finite-dimensional subspace F of X such that*

$$\text{Mid}(x, -x, \delta) \subset F + 2(p\delta)^{\frac{1}{p}} \|x\| B_X,$$

2. *there is a finite-codimensional subspace Y of X such that*

$$Y \cap \delta^{\frac{1}{p}} \|x\| B_X \subset \text{Mid}(x, -x, \delta).$$

Proof: Let $0 < \delta < 1$, let $x \in X$, and let $\epsilon > 0$. We can uniquely write $x = \sum_{n=1}^{\infty} x_n$,

with $x_n \in F_n$ for every n and $\|x\|^p = \sum_{n=1}^{\infty} \|x_n\|^p$.

Let N be such that $\sum_{n=1}^N \|x_n\|^p \geq (1 - \epsilon) \|x\|^p$, and hence $\sum_{n>N} \|x_n\|^p \leq \epsilon \|x\|^p$. Set $F = \overline{\text{span}}\{F_n, n = 1, \dots, N\}$ and $Y = \overline{\text{span}}\{F_n, n > N\}$. And for every $y \in X$, write y uniquely as $y = y_F + y_Y$ with $y_F \in F$ and $y_Y \in Y$.

1. Let $y = y_F + y_Y \in \text{Mid}(x, -x, \delta)$. By virtue of the triangle inequality, we cannot both have $\|x_F + y_F\| < \|x_F\|$ and $\|x_F - y_F\| < \|x_F\|$. So let us assume without loss of generality that $\|x_F + y_F\| \geq \|x_F\|$.

Since $\|x + y\| \leq (1 + \delta)\|x\|$, we get:

$$\begin{aligned}
 (1 + \delta)^p \|x\|^p &\geq \|x + y\|^p \\
 &= \|x_F + y_F\|^p + \|x_Y + y_Y\|^p \\
 &\geq \|x_F\|^p + \|x_Y + y_Y\|^p \\
 &\geq (1 - \epsilon)\|x\|^p + \|x_Y + y_Y\|^p.
 \end{aligned}$$

This implies that

$$\|x_Y + y_Y\| \leq ((1 + \delta)^p - (1 - \epsilon))^{\frac{1}{p}} \|x\|$$

and hence

$$\|y_Y\| \leq \left(((1 + \delta)^p - (1 - \epsilon))^{\frac{1}{p}} + \epsilon^{\frac{1}{p}} \right) \|x\|.$$

Now, since $0 < \delta < 1$, we have that $(1 + \delta)^p \leq 1 + 2^{p-1}p\delta$, and hence we have

$$\left(((1 + \delta)^p - (1 - \epsilon))^{\frac{1}{p}} + \epsilon^{\frac{1}{p}} \right) \leq (2^{p-1}p\delta + \epsilon)^{\frac{1}{p}} + \epsilon^{\frac{1}{p}},$$

and this is smaller than $2(p\delta)^{\frac{1}{p}}$ if ϵ is chosen to be small enough.

2. Let $y \in Y$ where Y is the finite-codimensional subspace described above. Assume $\|y\| \leq \delta^{\frac{1}{p}}\|x\|$. We have:

$$\begin{aligned}
 \|x_F \pm y\|^p &= \|x_F\|^p + \|y\|^p \\
 &\leq \|x_F\|^p + \delta\|x\|^p.
 \end{aligned}$$

Let us consider first the case $p = 1$. From the above we have

$$\|x_F \pm y\| \leq \|x_F\| + \delta\|x\|,$$

and hence by triangle inequality,

$$\|x \pm y\| \leq \|x_F\| + \delta\|x\| + \|x_Y\| = (1 + \delta)\|x\|.$$

For the case $p > 1$, let us use the trivial upper bound that $\|x_F\| \leq \|x\|$ to get

$$\|x_F \pm y\|^p \leq (1 + \delta)\|x\|^p$$

$$\|x_F \pm y\| \leq (1 + \delta)^{\frac{1}{p}}\|x\|.$$

So

$$\begin{aligned} \|x \pm y\| &\leq (1 + \delta)^{\frac{1}{p}}\|x\| + \|x_Y\| \\ &\leq \left((1 + \delta)^{\frac{1}{p}} + \epsilon^{\frac{1}{p}} \right) \|x\|. \end{aligned}$$

When ϵ is small enough, we have $\left((1 + \delta)^{\frac{1}{p}} + \epsilon^{\frac{1}{p}} \right) \leq (1 + \delta)$. □

The idea of “comparing the sizes” of approximate midpoints in the study of the uniform classification of Banach spaces is due to Enflo (as is mentioned in [BeL, page 254]) when he proved that L_1 and ℓ_1 are not uniformly homeomorphic. The main ingredient in being able to do this comparison is expressed in the following lemma (see [BeL, Lemma 10.11]).

Lemma 2.4.3 (Stretching Lemma). *Let X and Y be Banach spaces and let $f : X \rightarrow Y$ be uniformly continuous. Call K_d the Lipschitz constant of f for distances larger than d , i.e. $\|f(x) - f(y)\| \leq K_d\|x - y\|$ for every $x, y \in X$ such that $\|x - y\| \geq d$. Define $K_\infty = \lim_{d \rightarrow \infty} K_d$. Assume $K_\infty > 0$. Then for every $0 < \delta < 1/2$ and for every $D > 0$ there are points $x, y \in X$ with $\|x - y\| \geq D$ such that*

$$f(\text{Mid}(x, y, \delta)) \subset \text{Mid}(f(x), f(y), 5\delta).$$

Proof: Let $\delta \in (0, 1/2)$, and let $d > 0$ be large enough so that $\frac{K_d}{K_d} \leq 1 + \delta$. Let $x, y \in X$ with $\|x - y\| \geq d$ be such that $\|f(x) - f(y)\| \geq \frac{K_d}{1 + \delta} \|x - y\|$, and let $z \in \text{Mid}(x, y, \delta)$. Then by the triangle inequality we have that $\|x - z\| \geq (1 - \delta) \frac{\|x - y\|}{2} \geq \frac{d}{4}$ and hence:

$$\begin{aligned}
\|f(x) - f(z)\| &\leq K_{\frac{d}{4}} \|x - z\| \\
&\leq K_{\frac{d}{4}} (1 + \delta) \frac{\|x - y\|}{2} \\
&\leq \frac{K_d}{K_d} (1 + \delta)^2 \frac{\|f(x) - f(y)\|}{2} \\
&\leq (1 + \delta)^3 \frac{\|f(x) - f(y)\|}{2} \\
&\leq (1 + 5\delta) \frac{\|f(x) - f(y)\|}{2}.
\end{aligned}$$

□

The previous lemma says that pairs of points that “stretch” the function f , i.e. at which f almost attains its Lipschitz constant, are such that their approximate midpoints are sent into approximate midpoints of their images, and hence by comparing the sizes of approximate midpoints we get the following result [JoLS]:

Proposition 2.4.4. *Let $1 \leq p < r < +\infty$. Suppose that X is an ℓ_r -sum of the finite-dimensional spaces $(E_n)_n$, and Y is an ℓ_p -sum of the finite-dimensional spaces $(F_n)_n$. Then it is impossible that there be a uniform embedding of X into Y whose inverse is colipschitz for large distances.*

Proof: Say $j : X \rightarrow Y$ is a uniform embedding. Then we can find positive constants K and L such that whenever $\|x - y\| \geq 1$ we have:

$$L\|x - y\| \leq \|j(x) - j(y)\| \leq K\|x - y\|$$

Fix $0 < \delta < 1/2$ and choose d such that $\delta^{1/r} d > 2$. By the Stretching Lemma (Lemma 2.4.3), and via possible translations, we can find a point $x \in X$ with $\|x\| \geq d$

for which

$$j(\text{Mid}(x, -x, \delta)) \subset \text{Mid}(j(x), -j(x), 5\delta).$$

By the second part of Proposition 2.4.2 for X we can find an infinite sequence $(y_k)_k \subset \text{Mid}(x, -x, \delta)$ such that $\|y_k - y_l\| \geq \frac{1}{2}\delta^{1/r}\|x\|$. Thus $\|y_k - y_l\| \geq 1$ by the choice of d , and $\|j(y_k) - j(y_l)\| \geq L\|y_k - y_l\| \geq \frac{L}{2}\delta^{1/r}\|x\|$. The first part of Proposition 2.4.2 for Y , and the finite-dimensionality of F give that there are $k \neq l$ with $\|j(y_k) - j(y_l)\| \leq 5(5p\delta)^{1/p}\|j(x)\| \leq 5(5p\delta)^{1/p}K\|x\|$.

Thus $5(5p\delta)^{1/p}K \geq \frac{L}{2}\delta^{1/r}$, which is impossible when δ is small enough because $p < r$. □

We now come to the main result of this chapter.

2.5 Main result

Theorem 2.5.1. *Let $1 < p < 2$, and let Y be a linear quotient of ℓ_p and $u : Y \rightarrow X$ be a uniform homeomorphism. Then X is also a linear quotient of ℓ_p .*

Proof: By the result of [BJLPS] stated in Corollary 2.2.7, we know that X is a linear quotient of L_p , and so in particular it is reflexive. By dualizing the result in Theorem 2.3.1, we only need to show that ℓ_2 is not a linear quotient of X . By way of contradiction, let us assume that is the case, and let $Q : X \rightarrow \ell_2$ be a linear quotient map.

By taking adjoints, we get that ℓ_2 embeds into X^* which then embeds into L_{p^*} where $\frac{1}{p} + \frac{1}{p^*} = 1$ (so in particular $p^* > 2$). We can use Maurey's Extension Theorem (Theorem 2.3.2) to get that ℓ_2 is complemented in X^* by a projection say $P : X^* \rightarrow \ell_2$. By taking the adjoint of P again, we get that ℓ_2 embeds into X .

On the other hand, Y is a linear quotient of ℓ_p , so by Theorem 2.3.3 Y embeds linearly into an ℓ_p -sum of finite-dimensional spaces, say by an embedding $I : Y \rightarrow$

$(\sum_k F_k)_p$. So $I \circ u^{-1} \circ P^*$ is a uniform embedding of ℓ_2 into $(\sum_k F_k)_p$ which is co-Lipschitz for large distances, and we know from Proposition 2.4.4 that this is impossible since $p < 2$. \square

CHAPTER III

BANACH SPACES CONTAINING A LIPSCHITZ COPY OF c_0

In this chapter, we are going to generalize the method we used in previous chapter via the use of two moduli introduced by Milman in 1971 [Mi]. These moduli were later investigated by Johnson, Lindenstrauss, Preiss and Schechtman in [JoLPS] and they called them moduli of asymptotic uniform convexity and of asymptotic uniform smoothness. We will recall the definitions and properties of these moduli in the first half of the chapter. In the second half, we will tie them to the method presented in the previous chapter. Throughout, we will restrict our attention to real Banach spaces.

3.1 Modulus of asymptotic uniform convexity

Definition 3.1.1. Let X be a Banach space. The *modulus of asymptotic uniform convexity* of X is the map $\bar{\delta}_X$ defined on the interval $[0, 1]$ by

$$\bar{\delta}_X(t) = \inf_{x \in S_X} \sup_{Y \in \text{cof}(X)} \inf_{\substack{y \in Y \\ \|y\| \geq t}} \|x + y\| - 1,$$

where S_X is the unit sphere of X and $\text{cof}(X)$ is the family of all finite codimensional subspaces of X .

The Banach space X is called *asymptotically uniformly convex* if $\bar{\delta}_X(t) > 0$ for all $0 < t \leq 1$.

This notion generalizes that of uniform convexity as is pointed out by the next Proposition [JoLPS]:

Proposition 3.1.2. *If X is uniformly convex, then X is asymptotically uniformly convex.*

Proof: Let $0 < t \leq 1$ and fix $\epsilon > 0$. Denote by δ_X the modulus of uniform convexity of X , i.e.

$$\delta_X(t) = \inf_{\substack{x, y \in B_X \\ \|x - y\| \geq t}} 1 - \left\| \frac{x + y}{2} \right\|.$$

Since X is uniformly convex, we have $\delta_X(t) > 0$.

Let $x \in S_X$, let x^* be a norming functional for x , i.e. $x^* \in S_{X^*}$ and $x^*(x) = \|x\|$.

Denote $Y = \ker(x^*)$, and denote

$$\bar{\delta}_X(x, Y, t) = \inf_{\substack{y \in Y \\ \|y\| \geq t}} \|x + y\| - 1.$$

It suffices to show that $\delta_X(t) \leq \bar{\delta}_X(x, Y, t) + \epsilon$.

First of all, let us notice that

$$\bar{\delta}_X(x, Y, t) = \inf_{\substack{y \in Y \\ \|y\| = t}} \|x + y\| - 1.$$

In fact, if we fix $y \in S_Y$ and denote $\varphi(t) = \|x + ty\| - 1$, then since $Y = \ker(x^*)$ we see that $\varphi(t) \geq x^*(x + ty) - 1 = 0$, i.e. φ attains its minimum at $t = 0$. Since φ is a convex function and $\varphi(t) \rightarrow \infty$ as $t \rightarrow \infty$, we notice that φ is a nondecreasing function.

Now, let $y_0 \in Y$ be such that $\|y_0\| = t$ and $1 + \bar{\delta}_X(x, Y, t) + \epsilon > \|x + y_0\|$. Call $a = \frac{x + y_0}{\|x + y_0\|}$ and $b = a - y_0$. Then $\|a - b\| = t$, $\|a\| = 1$, and $b = \frac{1}{\|x + y_0\|} x + \left(1 - \frac{1}{\|x + y_0\|}\right) (-y_0)$ is a convex combination of two elements of B_X , so $\|b\| \leq 1$. So we have:

$$\begin{aligned}
\delta_X(t) &\leq 1 - \frac{\|a+b\|}{2} \\
&\leq 1 - x^* \left(\frac{a+b}{2} \right) \\
&\leq 1 - \frac{1}{\|x+y_0\|} \\
&\leq 1 - \frac{1}{1 + \bar{\delta}_X(x, Y, t) + \epsilon} \\
&\leq \bar{\delta}_X(x, Y, t) + \epsilon.
\end{aligned}$$

□

So uniformly convex spaces are asymptotically uniformly convex. Notice however that the notion of asymptotic uniform convexity is a strict generalization of that of uniform convexity as we shall see next.

Proposition 3.1.3 (Modulus of AUC of ℓ_1). *For every $t \in [0, 1]$, $\bar{\delta}_{\ell_1}(t) = t$.*

Proof: As in the previous proof, notice that we have for any Banach space X

$$\bar{\delta}_X(t) = \inf_{x \in S_X} \sup_{Y \in \text{cof}(X)} \inf_{\substack{y \in Y \\ \|y\|=t}} \|x+y\| - 1,$$

and hence by the triangle inequality we always have $\bar{\delta}_X(t) \leq t$.

Conversely, let $x = (x_n)_n \in \ell_1$ with $\|x\| = \sum_{n=1}^{\infty} |x_n| = 1$. Fix $\epsilon > 0$ and let N be such that $\sum_{n=1}^N |x_n| > 1 - \epsilon$. Denote by Y_0 the finite codimensional subspace of ℓ_1 consisting of those vectors supported only on $\{n \in \mathbb{N}, n > N\}$, and let $y \in Y_0$ with $\|y\| \geq t$. We have:

$$\begin{aligned}
\|x + y\| &= \sum_{n=1}^N |x_n| + \sum_{n=N+1}^{\infty} |x_n + y_n| \\
&\geq 1 - \epsilon + \sum_{n=N+1}^{\infty} (|y_n| - |x_n|) \\
&\geq 1 - 2\epsilon + \|y\| \\
&\geq 1 - 2\epsilon + t.
\end{aligned}$$

Hence we have $\inf_{\substack{y \in Y_0 \\ \|y\| \geq t}} \|x + y\| - 1 \geq t - 2\epsilon$.

Taking supremum over all finite codimensional subspaces, and then infimum over all unit vectors, we then get $\bar{\delta}_{\ell_1}(t) \geq t - 2\epsilon$. Since ϵ is arbitrary, we have the result. \square

As we can see, ℓ_1 is a nonreflexive Banach space which is asymptotically uniformly convex. Moreover, since for any Banach space X we have $\bar{\delta}_X(t) \leq t$, we see that ℓ_1 attains the best modulus of asymptotic uniform convexity. So we can think of ℓ_1 as the most asymptotically uniformly convex space.

3.2 Modulus of asymptotic uniform smoothness

Definition 3.2.1. The *modulus of asymptotic uniform smoothness* of a Banach space X is the map $\bar{\rho}_X$ defined on the interval $[0, 1]$ by

$$\bar{\rho}_X(t) = \sup_{x \in S_X} \inf_{Y \in \text{cof}(X)} \sup_{\substack{y \in Y \\ \|y\| \leq t}} \|x + y\| - 1,$$

where again S_X is the unit sphere of X and $\text{cof}(X)$ is the family of all finite codimensional subspaces of X .

The Banach space X is called *asymptotically uniformly smooth* if $\frac{\bar{\rho}_X(t)}{t} \rightarrow 0$ as $t \rightarrow 0$.

Proposition 3.2.2. *If X is uniformly smooth, then X is asymptotically uniformly smooth.*

Proof: Recall that the modulus of uniform smoothness ρ_X of X is

$$\rho_X(t) = \sup_{\substack{\|x\|=1 \\ \|y\|\leq t}} \frac{\|x+y\| + \|x-y\|}{2} - 1,$$

and that $\frac{\rho_X(t)}{t} \rightarrow 0$ as $t \rightarrow 0$ since X is uniformly smooth.

Let $x \in S_X$, let $x^* \in S_{X^*}$ be a norming functional for x and set $Y_0 = \ker(x^*)$.

For each $y \in Y_0$ with $\|y\| \leq t$ we have

$$\begin{aligned} \frac{1}{2}(\|x+y\| - 1) &= \frac{\|x+y\| + 1}{2} - 1 \\ &\leq \frac{\|x+y\| + \|x-y\|}{2} - 1 \\ &\leq \rho_X(t) \end{aligned}$$

because $\|x-y\| \geq x^*(x-y) = 1$.

Hence by taking supremum over all $y \in Y_0$ with $\|y\| \leq t$, and then infimum over all finite codimensional subspaces, and then supremum over all $x \in S_X$, we get

$$\frac{1}{2}\bar{\rho}_X(t) \leq \rho_X(t).$$

On the other hand, for every $y \in Y_0$ we have $\|x+y\| - 1 \geq x^*(x+y) - 1 = 0$,

so for any finite codimensional subspace Y of X we have

$$\sup_{\substack{y \in Y \cap Y_0 \\ \|y\| \leq t}} \|x+y\| - 1 \geq 0.$$

Hence we also have

$$\sup_{\substack{y \in Y \\ \|y\| \leq t}} \|x+y\| - 1 \geq 0,$$

and thus $\bar{\rho}_X(t) \geq 0$. □

Proposition 3.2.3 (Modulus of AUS of c_0). *The Banach space c_0 is asymptotically uniformly flat, namely $\bar{\rho}_{c_0}(1) = 0$.*

Proof: From the previous proof, we always have for any Banach space X that $\bar{\rho}_X(t) \geq 0$ for any $t \in [0, 1]$.

Conversely, let $x = (x_n)_n \in c_0$ with $\|x\| = \sup\{|x_n|, n \geq 1\} = 1$. Fix $0 < \epsilon < 1$, and let $N > 1$ be such that $|x_n| < \epsilon$ for all $n > N$. Hence in particular $\max\{|x_n|, 1 \leq n \leq N\} = 1$. Call Y_0 the finite codimensional subspace of c_0 consisting of those vectors supported on $\{n \in \mathbb{N}, n > N\}$, and let $y \in Y_0$ be such that $\|y\| \leq 1$. We have:

$$\begin{aligned}
 \|x + y\| &= \sup_n |x_n + y_n| \\
 &= \max\left\{\sup_{n \leq N} |x_n|, \sup_{n > N} |x_n + y_n|\right\} \\
 &\leq \max\left\{1, \sup_{n > N} (|x_n| + |y_n|)\right\} \\
 &\leq \max\left\{1, \sup_{n > N} |x_n| + \sup_{n > N} |y_n|\right\} \\
 &\leq \max\{1, 1 + \epsilon\} \\
 &\leq 1 + \epsilon.
 \end{aligned}$$

Taking supremum, infimum, supremum as is now customary, we get that $\bar{\rho}_{c_0}(1) \leq \epsilon$. □

3.3 Properties of the asymptotic moduli

The following are properties of the moduli $\bar{\delta}_X$ and $\bar{\rho}_X$ that come directly from their definitions [JoLPS]:

Proposition 3.3.1. *For a Banach space X ,*

1. $\bar{\delta}_X$ and $\bar{\rho}_X$ are nondecreasing.
2. for each $t \in [0, 1]$, $0 \leq \bar{\delta}_X(t) \leq \bar{\rho}_X(t) \leq t$.

3. if Y is an infinite-dimensional subspace of X , then $\bar{\delta}_X \leq \bar{\delta}_Y$ and $\bar{\rho}_Y \leq \bar{\rho}_X$.

Proof:

1. Directly from the definitions.
2. We have practically seen in the previous proofs that $\bar{\delta}_X(t) \geq 0$ and $\bar{\rho}_X(t) \leq t$.

For the middle inequality, let $x \in S_X$ and denote

$$\begin{aligned}\bar{\delta}_{X,x}(t) &= \sup_{Y \in \text{cof}(x)} \inf_{\substack{y \in Y \\ \|y\| \geq t}} \|x + y\| - 1, \\ \bar{\rho}_{X,x}(t) &= \inf_{Y \in \text{cof}(x)} \sup_{\substack{y \in Y \\ \|y\| \leq t}} \|x + y\| - 1.\end{aligned}$$

Let $\epsilon > 0$, and let Y_0 and Y_1 be finite codimensional subspaces such that

$$\begin{aligned}\inf_{\substack{y \in Y_0 \\ \|y\| \geq t}} \|x + y\| - 1 &> \bar{\delta}_{X,x}(t) - \epsilon, \\ \sup_{\substack{y \in Y_1 \\ \|y\| \leq t}} \|x + y\| - 1 &< \bar{\rho}_{X,x}(t) + \epsilon.\end{aligned}$$

Then $Y_0 \cap Y_1$ is also a finite codimensional subspace of X and

$$\inf_{\substack{y \in Y_0 \cap Y_1 \\ \|y\| = t}} \|x + y\| - 1 \leq \sup_{\substack{y \in Y_0 \cap Y_1 \\ \|y\| = t}} \|x + y\| - 1.$$

In this last inequality, we have

$$\begin{aligned}\text{RHS} &\leq \sup_{\substack{y \in Y_0 \cap Y_1 \\ \|y\| \leq t}} \|x + y\| - 1 \\ &\leq \sup_{\substack{y \in Y_1 \\ \|y\| \leq t}} \|x + y\| - 1 \\ &< \bar{\rho}_{X,x}(t) + \epsilon,\end{aligned}$$

and similarly

$$\text{LHS} > \bar{\delta}_{X,x}(t) - \epsilon.$$

Hence we always have $\bar{\delta}_{X,x}(t) \leq \bar{\rho}_{X,x}(t)$ and the result follows immediately from here.

3. For a subspace Y of X , we have $\{Z \cap Y, Z \in \text{cof}(X)\} \subset \text{cof}(Y)$. Hence:

$$\begin{aligned}
\bar{\delta}_Y(t) &= \inf_{y \in S_Y} \sup_{W \in \text{cof}(Y)} \inf_{\substack{w \in W \\ \|w\| \geq t}} \|y + w\| - 1 \\
&\geq \inf_{y \in S_Y} \sup_{Z \in \text{cof}(X)} \inf_{\substack{z \in Z \cap Y \\ \|z\| \geq t}} \|y + z\| - 1 \\
&\geq \inf_{y \in S_Y} \sup_{Z \in \text{cof}(X)} \inf_{\substack{z \in Z \\ \|z\| \geq t}} \|y + z\| - 1 \\
&\geq \inf_{x \in S_X} \sup_{Z \in \text{cof}(X)} \inf_{\substack{z \in Z \\ \|z\| \geq t}} \|x + z\| - 1 \\
&= \bar{\delta}_X(t).
\end{aligned}$$

In a similar manner we also get $\bar{\rho}_Y(t) \leq \bar{\rho}_X(t)$. \square

Remark 3.3.2. From the previous Proposition we see that if Y is a subspace of X , then we have $\bar{\delta}_Y(t) \leq \bar{\rho}_X(t)$. The purpose of this chapter is to show that we still have such an inequality up to some constant if X is just assumed to contain a Lipschitz copy of Y .

3.4 Asymptotic moduli and approximate midpoints

In this section, we will relate the sizes of the approximate metric midpoints of points in the Banach space X with the moduli of asymptotic uniform convexity and smoothness of X .

Proposition 3.4.1 (Modulus of AUS and approximate midpoints). *Let X be a Banach space. Let $t \in [0, 1]$, let $\epsilon > 0$ and let $x \in X$. Then one can find a finite codimensional subspace Y of X such that*

$$t\|x\|B_Y \subset \text{Mid}(x, -x, \bar{\rho}_X(t) + \epsilon).$$

Proof: Without loss of generality, we can assume that $\|x\| = 1$. Since $\epsilon > 0$, we have:

$$\sup_{u \in S_X} \inf_{Y \in \text{cof}(X)} \sup_{\substack{y \in Y \\ \|y\| \leq t}} \|u + y\| < 1 + \bar{\rho}_X(t) + \epsilon,$$

so in particular,

$$\inf_{Y \in \text{cof}(X)} \sup_{\substack{y \in Y \\ \|y\| \leq t}} \|x + y\| < 1 + \bar{\rho}_X(t) + \epsilon.$$

Hence we can find a finite codimensional subspace Y_0 of X such that

$$\sup_{\substack{y \in Y_0 \\ \|y\| \leq t}} \|x + y\| < 1 + \bar{\rho}_X(t) + \epsilon,$$

or in other words we have for every $y \in tB_{Y_0}$

$$\|x + y\| < 1 + \bar{\rho}_X(t) + \epsilon.$$

By the symmetry of the ball B_{Y_0} , we also then have

$$\|x - y\| < 1 + \bar{\rho}_X(t) + \epsilon.$$

This means that

$$tB_{Y_0} \subset \text{Mid}(x, -x, \bar{\rho}_X(t) + \epsilon).$$

□

Proposition 3.4.2 (Modulus of AUC and approximate midpoints). *Let X be a real Banach space that is asymptotically uniformly convex. Let $t \in (0, 1]$, let $0 < \epsilon < \bar{\delta}_X(t)$ and let $x \in X$. Then one can find a compact subset K of X such that*

$$\text{Mid}(x, -x, \bar{\delta}_X(t) - \epsilon) \subset K + 3t\|x\|B_X.$$

Proof: Again without loss of generality, we may assume that $\|x\| = 1$. Since $\epsilon > 0$, we have:

$$\inf_{u \in S_X} \sup_{Y \in \text{cof}(X)} \inf_{\substack{y \in Y \\ \|y\| \geq t}} \|u + y\| > 1 + \bar{\delta}_X(t) - \epsilon,$$

so in particular,

$$\sup_{Y \in \text{cof}(X)} \inf_{\substack{y \in Y \\ \|y\| \geq t}} \|x + y\| > 1 + \bar{\delta}_X(t) - \epsilon.$$

Hence we can find a finite codimensional subspace Y_0 of X such that

$$\inf_{\substack{y \in Y_0 \\ \|y\| \geq t}} \|x + y\| > 1 + \bar{\delta}_X(t) - \epsilon,$$

or in other words we have for every $y \in Y_0$ with $\|y\| \geq t$,

$$\|x + y\| > 1 + \bar{\delta}_X(t) - \epsilon.$$

Since $-y \in Y_0$ and $\| -y \| \geq t$, we also then have

$$\|x - y\| > 1 + \bar{\delta}_X(t) - \epsilon.$$

By contraposition, this means that if $y \in Y_0$ and at least one of $\|x + y\|$ and $\|x - y\|$ is less than or equal to $1 + \bar{\delta}_X(t) - \epsilon$, then $\|y\| < t$. We then have the weaker statement:

$$Y_0 \cap \text{Mid}(x, -x, \bar{\delta}_X(t) - \epsilon) \subset tB_X.$$

Denote $C = \text{Mid}(x, -x, \bar{\delta}_X(t) - \epsilon)$. Then since $\epsilon < \bar{\delta}_X(t)$, we notice that C is a convex closed symmetric bounded subset of X that contains a neighborhood of 0. So since we are working in a real Banach space, the Minkowski functional μ_C of C defines an equivalent norm on X whose unit ball is C . In particular, (X, μ_C) is a Banach space. We have an easy lemma (see e.g. [JoLPS]):

Lemma 3.4.3. *Let X be a Banach space, and let Y be a finite codimensional subspace of X . Then one can find a compact subset K of X such that*

$$B_X \subset K + 3B_Y.$$

Proof: Denote by $Q : X \rightarrow X/Y$ the canonical quotient map, and let $\epsilon = \frac{1}{4}$. We have $\frac{2+\epsilon}{1-\epsilon} = 3$. Since X/Y is finite dimensional, we can find an ϵ -net $(e_i)_{i=1,\dots,n}$ of $B_{X/Y}$. We can choose $\|e_i\| < 1$ for each $i \in \{1, \dots, n\}$, and so can choose points $x_i \in B_X$ such that $Q(x_i) = e_i$. Set $E = \text{span}\{x_i, i = 1, \dots, n\}$.

Let $x \in B_X$, and pick $i_0 \in \{1, \dots, n\}$ such that $\|Q(x) - e_{i_0}\| < \epsilon$, i.e. $d(x - x_{i_0}, Y) < \epsilon$. Pick $y \in Y$ such that $\|x - x_{i_0} - y\| < \epsilon$. Then $\|y\| \leq \epsilon + \|x - x_{i_0}\| \leq 2 + \epsilon$. As a result, we have written x as an element of $B_E + (2 + \epsilon)B_Y + \epsilon B_X$.

Now we have

$$B_X \subset B_E + (2 + \epsilon)B_Y + \epsilon B_X,$$

so by induction we get for every $k \geq 1$

$$B_X \subset (1 + \epsilon + \dots + \epsilon^k)(B_E + (2 + \epsilon)B_Y) + \epsilon^{k+1}B_X,$$

i.e. for every $k \geq 1$

$$B_X \subset \left(\frac{1}{1-\epsilon}\right)(B_E + (2 + \epsilon)B_Y) + \epsilon^{k+1}B_X.$$

Write $x \in B_X$ as $x = s_k + r_k$ with $s_k \in \left(\frac{1}{1-\epsilon}\right)(B_E + (2 + \epsilon)B_Y)$ and $r_k \in \epsilon^{k+1}B_X$. So $r_k \rightarrow 0$ and hence $s_k \rightarrow x$. Since B_E is compact, $\left(\frac{1}{1-\epsilon}\right)(B_E + (2 + \epsilon)B_Y)$ is closed and so $x \in \left(\frac{1}{1-\epsilon}\right)(B_E + (2 + \epsilon)B_Y)$.

Set $K = \left(\frac{1}{1-\epsilon}\right)B_E$ and we have our result $B_X \subset K + 3B_Y$. \square

Continuation of the proof of 3.4.2: Applying Lemma 3.4.3 to the Banach space (X, μ_C) and the finite codimensional subspace Y_0 , we can find a compact subset K of X such that

$$B_{X, \mu_C} \subset K + 3B_{Y_0, \mu_C}.$$

But $B_{X, \mu_C} = C$ and $B_{Y_0, \mu_C} = Y_0 \cap C$. And since

$$Y_0 \cap C = Y_0 \cap \text{Mid}(x, -x, \bar{\delta}_X(t) - \epsilon) \subset tB_X,$$

we get that

$$\text{Mid}(x, -x, \bar{\delta}_X(t) - \epsilon) \subset K + 3tB_X.$$

□

3.5 Main result

Theorem 3.5.1. *Let X and Y be real Banach spaces, and let $f : X \rightarrow Y$ be a uniform embedding which is co-Lipschitz for large distances. Assume that Y is asymptotically uniformly convex. Then there exists a constant $C \geq 1$ such that for each $t \in (0, 1/C]$ we have:*

$$\bar{\delta}_Y(t) \leq 10\bar{\rho}_X(Ct).$$

Proof: Let L be such that we have for all $x, x' \in X$ with $\|x - x'\| \geq 1$:

$$\frac{1}{L}\|x - x'\| \leq \|f(x) - f(x')\| \leq L\|x - x'\|.$$

Let $t \in (0, 1]$, and notice that $0 < \frac{\bar{\delta}_Y(t)}{10} < \frac{1}{2}$. Use the Stretching Lemma (Lemma 2.4.3) to find points x and x' in X with $\|x - x'\|$ as large as we please such that

$$f\left(\text{Mid}\left(x, x', \frac{\bar{\delta}_Y(t)}{10}\right)\right) \subset \text{Mid}\left(f(x), f(x'), \frac{\bar{\delta}_Y(t)}{2}\right).$$

Via possible translations, we can assume without loss of generality that $x' = -x$ and $f(x') = -f(x)$.

Use $\epsilon = \frac{\bar{\delta}_Y(t)}{2}$ in Proposition 3.4.2 to produce a compact subset K of Y such that

$$\text{Mid}\left(f(x), f(x'), \frac{\bar{\delta}_Y(t)}{2}\right) \subset K + 3t\|f(x)\|B_Y,$$

i.e.

$$\text{Mid}\left(f(x), f(x'), \frac{\bar{\delta}_Y(t)}{2}\right) \subset K + 3tL\|x\|B_Y.$$

Now, assume that we can find an $\epsilon > 0$ such that

$$\bar{\rho}_X(28L^2t) + \epsilon < \frac{\bar{\delta}_Y(t)}{10}.$$

Then

$$\text{Mid}\left(x, -x, \bar{\rho}_X(28L^2t) + \epsilon\right) \subset \text{Mid}\left(x, -x, \frac{\bar{\delta}_Y(t)}{10}\right),$$

and using Proposition 3.4.1 we can then find a finite codimensional subspace X_0 of X such that

$$f(28tL^2\|x\|B_{X_0}) \subset K + 3tL\|x\|B_Y.$$

Now, since X_0 is infinite dimensional, we can find an infinite sequence $(x_n)_n$ in $28tL^2\|x\|B_{X_0}$ such that $\|x_n - x_m\| > 14tL^2\|x\|$ for every $n \neq m$. Since $\|x\|$ was chosen to be large enough, we can use the co-Lipschitz condition on f for distances larger than 1 to get for every $n \neq m$

$$\|f(x_n) - f(x_m)\| > 14tL\|x\|.$$

Write $f(x_n)$ as $f(x_n) = k_n + y_n$ where $k_n \in K$ and $\|y_n\| \leq 3tL\|x\|$. Then since K is compact, we can assume (by possibly passing to a further subsequence) that for every $n \neq m$

$$\|k_n - k_m\| < 7tL\|x\|.$$

Hence

$$\begin{aligned} 6tL\|x\| &\geq \|y_n\| + \|y_m\| \\ &\geq \|y_n - y_m\| \\ &\geq \|f(x_n) - f(x_m)\| - \|k_n - k_m\| \\ &\geq 14tL\|x\| - 7tL\|x\|. \end{aligned}$$

This is a contradiction, and hence we must have

$$\frac{\bar{\delta}_Y(t)}{10} \leq \bar{\rho}_X(28L^2t).$$

So we can take $C = 28L^2$ and this finishes the proof. \square

Corollary 3.5.2. *Assume that c_0 Lipschitz embeds into a real Banach space Y . Then Y cannot be asymptotically uniformly convex under any renorming.*

Remark 3.5.3. A separable Banach space Y is called a Lipschitz universal space for separable metric spaces if every separable metric space admits a Lipschitz embedding into Y . It is well known that the Banach space $C[0,1]$ is a linear and therefore Lipschitz universal space. It was proved by Aharoni [A] that c_0 is Lipschitz universal for separable metric spaces. As a result, when we study a Banach space Y that is Lipschitz universal for separable metric spaces, it is enough to assume that c_0 Lipschitz embeds into Y .

CHAPTER IV

COARSE EMBEDDING INTO A HILBERT SPACE

In these last two chapters, we are going to concentrate on nonlinear maps that give a control on the distances when the distances are large. These maps are called coarse embeddings. The formal definition is given below (see [Gr1, 7.G]):

4.1 Coarse embeddings

Definition 4.1.1. A (not necessarily continuous) map f between two metric spaces (X, d) and (Y, δ) is called a *coarse embedding* if there exist two non-decreasing functions $\varphi_1 : [0, \infty) \rightarrow [0, \infty)$ and $\varphi_2 : [0, \infty) \rightarrow [0, \infty)$ such that

1. $\varphi_1(d(x, y)) \leq \delta(f(x), f(y)) \leq \varphi_2(d(x, y))$
2. $\varphi_1(t) \rightarrow \infty$ as $t \rightarrow \infty$.

We will be interested mainly in coarse embeddings of Banach spaces into a Hilbert space. One of the main ingredients we need for this study is Schoenberg's classical work on positive definite functions [S]. We will recall the definitions and the results of most importance to our study in the next section. This overview can also be seen in [BeL, page 185] with slightly different proofs. Again, all Banach spaces and Hilbert spaces considered in this chapter are real.

4.2 Negative definite kernels and related notions

Throughout this section, X will be an additive group, e.g. a Banach space.

Definition 4.2.1. A map $K : X \times X \rightarrow \mathbb{R}$ is called a *positive definite kernel* on X if

$$\forall x, y \in X, \quad K(x, y) = K(y, x),$$

and

$$\forall c_1, c_2, \dots, c_n \in \mathbb{R}, \forall x_1, x_2, \dots, x_n \in X, \sum_{i,j=1}^n c_i c_j K(x_i, x_j) \geq 0.$$

A map $f : X \rightarrow \mathbb{R}$ is called a *positive definite function* on X if the map $(x, y) \mapsto f(x - y)$ is a positive definite kernel.

A positive definite kernel K is called *normalized* if $K(x, x) = 1$ for each $x \in X$. Similarly, a positive definite function f is normalized if $f(0) = 1$.

Definition 4.2.2. A map $N : X \times X \rightarrow \mathbb{R}$ is called a *negative definite kernel* on X if

$$\forall x, y \in X, N(x, y) = N(y, x),$$

and

$$\forall x_1, x_2, \dots, x_n \in X, \forall c_1, c_2, \dots, c_n \in \mathbb{R}, \sum_{i=1}^n c_i = 0 \Rightarrow \sum_{i,j=1}^n c_i c_j N(x_i, x_j) \leq 0.$$

A map $g : X \rightarrow \mathbb{R}$ is called a *negative definite function* on X if the map $(x, y) \mapsto g(x - y)$ is a negative definite kernel.

A negative definite kernel N is called *normalized* if $N(x, x) = 0$ for each $x \in X$, and a negative definite function g is normalized if $g(0) = 0$.

Examples 4.2.3. Some examples of negative and positive definite kernels are the following:

- The constant kernel $(x, y) \mapsto 1$ is both a positive and a negative definite kernel.
- Let H be a Hilbert space. Let K and N be the kernels defined on H by $(x, y) \mapsto K(x, y) = \langle x, y \rangle$ and $(x, y) \mapsto N(x, y) = \|x - y\|^2$. Then K is a positive definite kernel, and N is a negative definite kernel. In fact:

1. Let $c_1, c_2, \dots, c_n \in \mathbb{R}$. We have:

$$\begin{aligned}
 \sum_{i,j=1}^n c_i c_j K(x_i, x_j) &= \sum_{i,j=1}^n c_i c_j \langle x_i, x_j \rangle \\
 &= \left\langle \sum_{i=1}^n c_i x_i, \sum_{j=1}^n c_j x_j \right\rangle \\
 &= \left\| \sum_{i=1}^n c_i x_i \right\|^2 \\
 &\geq 0.
 \end{aligned}$$

2. Now assume $\sum_{i=1}^n c_i = 0$. We have:

$$\begin{aligned}
 \sum_{i,j=1}^n c_i c_j N(x_i, x_j) &= \sum_{i,j=1}^n c_i c_j \|x_i - x_j\|^2 \\
 &= \sum_{i,j=1}^n c_i c_j (\|x_i\|^2 + \|x_j\|^2 - 2 \langle x_i, x_j \rangle) \\
 &= \left(\sum_{i=1}^n c_i \|x_i\|^2 \right) \left(\sum_{j=1}^n c_j \right) + \left(\sum_{j=1}^n c_j \|x_j\|^2 \right) \left(\sum_{i=1}^n c_i \right) \\
 &\quad - 2 \sum_{i,j=1}^n c_i c_j \langle x_i, x_j \rangle \\
 &= 0 + 0 - 2 \sum_{i,j=1}^n c_i c_j K(x_i, x_j) \\
 &\leq 0.
 \end{aligned}$$

One of the most important of Schoenberg's results that we will use here is the one showing that there are no other examples. More precisely, we have the following theorem.

Theorem 4.2.4 (Schoenberg's Theorem).

1. K is a positive definite kernel on X if and only if there exists a Hilbert space H and a map $T : X \rightarrow H$ such that $K(x, y) = \langle T(x), T(y) \rangle$ for every $x, y \in X$.

2. N is a normalized negative definite kernel on X if and only if there exists a Hilbert space H and a map $T : X \rightarrow H$ such that $N(x, y) = \|T(x) - T(y)\|^2$ for every $x, y \in X$.

Proof:

1. Let V_K be the linear vector space of all finitely supported real-valued maps on X , and define on V_K

$$\langle f, g \rangle = \sum_{x, y \in X} f(x)g(y)K(x, y).$$

Then $\langle \cdot, \cdot \rangle$ is an inner product on the quotient V_K/\mathcal{N}_K where $\mathcal{N}_K = \{f \in V_K, \langle f, f \rangle = 0\}$. Let H_K be the completion of V_K/\mathcal{N}_K , then H_K is a Hilbert space.

Define $T_K : X \rightarrow H_K$ by $T_K(x)$ being the equivalence class of the map δ_x where $\delta_x(y) = 1$ if $x = y$ and $\delta_x(y) = 0$ if $x \neq y$. We have:

$$\begin{aligned} K(x, y) &= \sum_{u, v \in X} \delta_x(u)\delta_y(v)K(u, v) \\ &= \langle \delta_x, \delta_y \rangle \\ &= \langle T_K(x), T_K(y) \rangle. \end{aligned}$$

2. Now let V_N be the linear vector space of all finitely supported real-valued maps f on X that satisfy $\sum_{x \in X} f(x) = 0$, and define on V_N

$$\langle f, g \rangle = -\frac{1}{2} \sum_{x, y \in X} f(x)g(y)N(x, y).$$

Then $\langle \cdot, \cdot \rangle$ is a semi-inner product. Let $\mathcal{N}_N = \{f \in V_N, \langle f, f \rangle = 0\}$, and let H_N be the completion of V_N/\mathcal{N}_N . Then H_N is a Hilbert space.

Define $T_N : X \rightarrow H_N$ by $T_N(x)$ being the equivalence class of the map δ_x where δ_x is defined as above. We have for every x and y :

$$\begin{aligned}\langle \delta_x, \delta_y \rangle &= -\frac{1}{2} \sum_{u,v} \delta_x(u) \delta_y(v) N(u, v) \\ &= \frac{1}{2} N(x, y).\end{aligned}$$

So $\|\delta_x\|^2 = \|\delta_y\|^2 = 0$ since N is normalized, and hence

$$\|T_N(x) - T_N(y)\|^2 = \|\delta_x - \delta_y\|^2 = -2\langle \delta_x, \delta_y \rangle = N(x, y).$$

□

The relationship between negative and positive definite kernels is expressed in the following lemma:

Lemma 4.2.5 (Schoenberg's Lemma). *The kernel N is negative definite if and only if the kernel e^{-tN} is positive definite for every $t > 0$.*

Proof: Assume that e^{-tN} is positive definite for every $t > 0$. Then $-e^{-tN}$ is negative definite. Since the constant kernel $(x, y) \mapsto 1$ is negative definite, we also have that $\frac{1 - e^{-tN}}{t}$ is negative definite. Since we have

$$N = \lim_{t \rightarrow 0} \frac{1 - e^{-tN}}{t},$$

we get that N is a negative definite kernel.

For the converse, we can assume without loss of generality that $t = \frac{1}{2}$, i.e we want to show that if N is negative definite then $e^{-\frac{N}{2}}$ is positive definite. By Schoenberg's Theorem, write $N(x, y) = \|T(x) - T(y)\|^2$ where T is a map from X to some Hilbert space H . So

$$N(x, y) = \|T(x)\|^2 + \|T(y)\|^2 - 2\langle T(x), T(y) \rangle.$$

Let $x_1, x_2, \dots, x_n \in X$ and let $c_1, c_2, \dots, c_n \in \mathbb{R}$. We have for every $i, j \in \{1, \dots, n\}$:

$$c_i c_j e^{-\frac{1}{2}N(x_i, x_j)} = c_i e^{-\frac{1}{2}\|T(x_i)\|^2} c_j e^{-\frac{1}{2}\|T(x_j)\|^2} e^{\langle T(x_i), T(x_j) \rangle}.$$

So by setting for every i , $c'_i = c_i e^{-\frac{1}{2}\|T(x_i)\|^2}$, it suffices to prove that the kernel $(x, y) \mapsto e^{\langle T(x), T(y) \rangle}$ is positive definite. Writing this in power series, it then suffices to prove that $(x, y) \mapsto \langle T(x), T(y) \rangle^k$ is positive definite for every $k \geq 0$. This we prove by induction.

Assume that for $k \geq 2$, the kernel $(x, y) \mapsto \langle T(x), T(y) \rangle^{k-1}$ is positive definite. Let $x_1, x_2, \dots, x_n \in X$ and let $c_1, c_2, \dots, c_n \in \mathbb{R}$. Since the kernel $(x, y) \mapsto \langle T(x), T(y) \rangle$ is positive definite, the matrix $(\langle T(x_i), T(x_j) \rangle)_{i,j=1,\dots,n}$ is a positive definite matrix. Hence we can find a matrix $(a_{ij})_{i,j}$ such that for each i and j , we have:

$$\langle T(x_i), T(x_j) \rangle = \sum_{l=1}^n a_{il} a_{jl},$$

and

$$\sum_{i,j=1}^n c_i c_j \langle T(x_i), T(x_j) \rangle^k = \sum_{l=1}^n \sum_{i,j=1}^n c_i a_{il} c_j a_{jl} \langle T(x_i), T(x_j) \rangle^{k-1}.$$

And this is nonnegative because the inner sum is always nonnegative from the induction hypothesis. \square

One last property that we will use is the following:

Proposition 4.2.6. *Let N be a negative definite kernel on X satisfying $N(x, y) \geq 0$ for every $x, y \in X$. Then for every $0 < \lambda < 1$, the kernel N^λ is also negative definite.*

Proof: To see this we use the classical identity

$$x^\lambda = c_\lambda \int_0^\infty \frac{1 - e^{-tx}}{t^{\lambda+1}} dt,$$

where c_λ is the constant

$$c_\lambda = \left(\int_0^\infty \frac{1 - e^{-u}}{u^{\lambda+1}} dt \right)^{-1}.$$

(Make the change of variable $u = tx$.)

So then

$$N^\lambda = c_\lambda \int_0^\infty \frac{1 - e^{-tN}}{t^{\lambda+1}} dt,$$

and the integrand is negative definite by applying Schoenberg's Lemma to N . Hence N^λ is also negative definite. \square

4.3 Extensions

In this section, we present another ingredient of importance to our study in this chapter, and that is the problem of extension. We have a metric space X and we have a map $f : S \rightarrow Y$ defined on a subset S of X into some metric space Y . This map f is assumed to be Lipschitz or α -Hölder for some λ , and the question is to study the structure of X and/or Y under which such a map can always be extended to the whole space X , the extension being also Lipschitz or α -Hölder respectively (with the same α).

Such problems have been widely studied, one of the latest results being that of Naor, Peres, Schramm, Sheffield [NPSS] about extending Lipschitz maps from a subset of ℓ_p ($p > 2$) to a Hilbert space. The result that we will use in this chapter is the following [WW, last statement of Theorem 19.1]:

Theorem 4.3.1. *Let X be a metric space, and let $0 < \alpha \leq \frac{1}{2}$. Let $f : S \rightarrow H$ be an α -Hölder map from a subset S of X to a Hilbert space H , i.e. there exists a constant C such that for every $x, y \in S$ one has:*

$$\|f(x) - f(y)\| \leq C d(x, y)^\alpha.$$

Then one can find a map $\tilde{f} : X \rightarrow Y$ such that \tilde{f} extends f and satisfying for every $x, y \in X$:

$$\|\tilde{f}(x) - \tilde{f}(y)\| \leq Cd(x, y)^\alpha.$$

4.4 Positive definite functions on ℓ_p , $p > 2$

Definition 4.4.1. Let X be a Banach space with a normalized Schauder basis $(e_n)_n$. Assume that $(e_n)_n$ is 1-symmetric, i.e. for any choice of signs $(\theta_n)_n \in \{-1, +1\}$ and any choice of permutation $\sigma : \mathbb{N} \rightarrow \mathbb{N}$,

$$\left\| \sum_n \theta_n a_n e_{\sigma(n)} \right\|_X = \left\| \sum_n a_n e_n \right\|_X.$$

A function $g : X \rightarrow \mathbb{R}$ is called *symmetric* if it satisfies for any choice of signs $(\theta_n)_n \in \{-1, +1\}$ and any choice of permutation $\sigma : \mathbb{N} \rightarrow \mathbb{N}$ the equality:

$$g \left(\sum_n \theta_n a_n e_{\sigma(n)} \right) = g \left(\sum_n a_n e_n \right).$$

Aharoni, Maurey and Mityagin proved the following theorem [AMM] which characterizes the symmetric continuous positive definite functions on a Banach space X with a symmetric basis behaving like that of ℓ_p , $p > 2$.

Theorem 4.4.2. *Let X be a Banach space with a symmetric basis $(e_i)_{i \geq 1}$. Assume that*

$$\liminf_{n \rightarrow \infty} \frac{\|e_1 + e_2 + \cdots + e_n\|}{\sqrt{n}} = 0.$$

Then every symmetric continuous positive definite function on X is constant.

4.5 Main result

The following is our main result:

Theorem 4.5.1. *Suppose that a Banach space X has a normalized symmetric basis $(e_n)_n$ and that $\liminf_{n \rightarrow \infty} n^{-\frac{1}{2}} \left\| \sum_{i=1}^n e_i \right\| = 0$. Then X does not coarsely embed into a Hilbert space.*

We present the proof in five steps.

Step 4.5.2 (Reducing to the α -Hölder case). Let $f : X \rightarrow H$ be a coarse embedding satisfying

1. $\varphi_1(\|x - y\|) \leq \|f(x) - f(y)\| \leq \varphi_2(\|x - y\|)$
2. $\varphi_1(t) \rightarrow \infty$ as $t \rightarrow \infty$.

Our first claim is that we do not lose generality by assuming that $\varphi_2(t) = t^\alpha$ with $0 < \alpha \leq \frac{1}{2}$.

To prove this claim, note first that $(x, y) \mapsto \|f(x) - f(y)\|^2$ is a negative definite kernel on X . This follows from Schoenberg's Theorem (Theorem 4.2.4). So by Proposition 4.2.6, for any $0 < \alpha < 1$, $N(x, y) = \|f(x) - f(y)\|^{2\alpha}$ is also negative definite kernel which is normalized.

As a result, Schoenberg's Theorem again allows us to find a Hilbert space H_α and a function $f_\alpha : X \rightarrow H_\alpha$ such that $N(x, y) = \|f_\alpha(x) - f_\alpha(y)\|^2$.

On the other hand, since X , being a normed space, is (metrically) convex, the original function $f : X \rightarrow H$ is Lipschitz for large distances (same proof as 2.1.2). Consequently, without loss of generality, we can assume by rescaling that we have the following for $\|x - y\| \geq 1$:

$$\|f(x) - f(y)\|_H \leq \|x - y\|$$

and

$$(\varphi_1(\|x - y\|))^\alpha \leq \|f_\alpha(x) - f_\alpha(y)\|_{H_\alpha} \leq \|x - y\|^\alpha.$$

Now, let S be a 1-net in X (i.e. S is a maximal 1-separated subset of X). The restriction of f_α to S is α -Hölder, so if $0 < \alpha \leq \frac{1}{2}$, then we can extend f_α to an α -Hölder map \tilde{f}_α defined on the whole of X by applying Theorem 4.3.1:

$$\tilde{f}_\alpha : X \rightarrow H_\alpha$$

$$\forall x \in S, \tilde{f}_\alpha(x) = f_\alpha(x)$$

and

$$\forall x, y \in X, \|f_\alpha(x) - f_\alpha(y)\|_{H_\alpha} \leq \|x - y\|^\alpha.$$

Now, write $\tilde{\varphi}_1(t) = \inf\{\|\tilde{f}_\alpha(x) - \tilde{f}_\alpha(y)\|, \|x - y\| \geq t\}$, and let us make sure that $\tilde{\varphi}_1(t) \rightarrow \infty$ as $t \rightarrow \infty$. To this end, let $x, y \in X$ and find $x_S, y_S \in S$ such that $\|x - x_S\| < 1$ and $\|y - y_S\| < 1$. We have:

$$\|\tilde{f}_\alpha(x) - \tilde{f}_\alpha(y)\| \geq \|\tilde{f}_\alpha(x_S) - \tilde{f}_\alpha(y_S)\| - \|\tilde{f}_\alpha(x) - \tilde{f}_\alpha(x_S)\| - \|\tilde{f}_\alpha(y) - \tilde{f}_\alpha(y_S)\|$$

$$= \|f_\alpha(x_S) - f_\alpha(y_S)\| - \|\tilde{f}_\alpha(x) - \tilde{f}_\alpha(x_S)\| - \|\tilde{f}_\alpha(y) - \tilde{f}_\alpha(y_S)\|$$

$$\geq (\varphi_1(\|x_S - y_S\|))^\alpha - \|x - x_S\|^\alpha - \|y - y_S\|^\alpha$$

$$\geq (\varphi_1(\|x_S - y_S\|))^\alpha - 2$$

$$\rightarrow \infty \text{ as } \|x - y\| \rightarrow \infty.$$

This finishes the proof of our reduction to the case where f is α -Hölder and thus uniformly continuous. So from now on we will assume that our coarse embedding

is a map $f : X \rightarrow H$ satisfying the following for all $x, y \in X$:

$$\varphi_1(\|x - y\|) \leq \|f(x) - f(y)\| \leq \|x - y\|^\alpha$$

where $\varphi_1(t) \rightarrow \infty$ as $t \rightarrow \infty$.

Step 4.5.3 (Controlling the growth of the negative definite kernel). Set $N(x, y) = \|f(x) - f(y)\|^2$. Then N is a normalized (i.e. $N(x, x) = 0$) negative definite kernel on X . Now if we write $\phi_1(t) = (\varphi_1(t))^2$ and $\phi_2(t) = t^{2\alpha}$, then N satisfies:

$$\begin{cases} \phi_1(\|x - y\|) \leq N(x, y) \leq \phi_2(\|x - y\|), \\ \phi_1(t) \rightarrow \infty \text{ as } t \rightarrow \infty. \end{cases}$$

Step 4.5.4 (Reducing to a negative definite function). The argument in this paragraph comes from [AMM, Lemma 3.5.]. Let μ be an invariant mean on the bounded functions on X (see e.g. [BeL]). We can think of μ as a finitely additive translation invariant positive measure for which $\mu(X) = 1$. Define:

$$g(x) = \int_X N(y + x, y) d\mu(y)$$

Then we have the following for g :

- g is well-defined because the map $y \mapsto N(y + x, y)$ is bounded for each $x \in X$,
- $g(0) = \int_X N(y, y) d\mu(y) = 0$,

- For scalars $(c_i)_{1 \leq i \leq n}$ satisfying $\sum_{i=1}^n c_i = 0$, we have:

$$\begin{aligned}
\sum_{i,j=1}^n c_i c_j g(x_i - x_j) &= \sum_{i,j} c_i c_j \int_X N(y + x_i - x_j, y) d\mu(y) \\
&= \sum_{i,j=1}^n c_i c_j \int_X N(y + x_i, y + x_j) d\mu(y) \\
&= \int_X \left(\sum_{i,j=1}^n c_i c_j N(y + x_i, y + x_j) \right) d\mu(y) \\
&= \int_X (\leq 0) d\mu(y) \\
&\leq 0.
\end{aligned}$$

This is because μ is translation invariant, and N is negative definite. This shows that g is a negative definite function on X .

- Finally, since $\int_X d\mu(y) = 1$, we have:

$$\phi_1(\|x\|) \leq g(x) \leq \phi_2(\|x\|).$$

In summary, we have found a negative definite function g on X which satisfies $g(0) = 0$ and $\phi_1(\|x\|) \leq g(x) \leq \phi_2(\|x\|)$, where $\phi_1(t) \rightarrow \infty$ as $t \rightarrow \infty$.

Step 4.5.5 (Reducing to a continuous symmetric negative definite function). Let $(e_n)_n$ be the normalized symmetric basis for X . This means that for any choice of signs $(\theta_n)_n \in \{-1, +1\}$ and any choice of permutation $\sigma : \mathbb{N} \rightarrow \mathbb{N}$,

$$\left\| \sum_n \theta_n a_n e_{\sigma(n)} \right\|_X = \left\| \sum_n a_n e_n \right\|_X.$$

The purpose of this paragraph is to show that the negative definite function g we found in the previous paragraph can be chosen to be symmetric, i.e. to satisfy for any choice of signs $(\theta_n)_n \in \{-1, +1\}$ and any choice of permutation $\sigma : \mathbb{N} \rightarrow \mathbb{N}$ the equality:

$$g \left(\sum_n \theta_n a_n e_{\sigma(n)} \right) = g \left(\sum_n a_n e_n \right).$$

For $x = \sum_{n=1}^{\infty} x_n e_n \in X$, define $g_m(x)$ to be the average (i.e. arithmetic mean) of $g \left(\sum_{n=1}^{\infty} \theta_n x_n e_{\sigma(n)} \right)$ over all choices of signs θ and permutations σ with the restrictions that $\theta_n = 1$ for $n > m$ and $\sigma(n) = n$ for $n > m$.

It follows that for all such θ, σ , and for all $x = \sum_{n=1}^{\infty} x_n e_n \in X$,

$$g_m \left(\sum_{n=1}^{\infty} \theta_n x_n e_{\sigma(n)} \right) = g_m \left(\sum_{n=1}^{\infty} x_n e_n \right).$$

Moreover, we also have

$$\phi_1(\|x\|) \leq g_m(x) \leq \phi_2(\|x\|).$$

Next we show that the sequence $(g_m)_m$ is equicontinuous. To check this, let us first check the continuity of g :

$$\begin{aligned}
|g(a) - g(b)| &\leq \int_X |N(y+a, y) - N(y+b, y)| \, d\mu(y) \\
&= \int_X \left| \|f(y+a) - f(y)\|^2 - \|f(y+b) - f(y)\|^2 \right| \, d\mu(y) \\
&= \int_X (\|f(y+a) - f(y)\| + \|f(y+b) - f(y)\|) \cdot \\
&\quad \left| \|f(y+a) - f(y)\| - \|f(y+b) - f(y)\| \right| \, d\mu(y) \\
&\leq \int_X (\|f(y+a) - f(y)\| + \|f(y+b) - f(y)\|) \cdot \\
&\quad \|f(y+a) - f(y+b)\| \, d\mu(y) \\
&\leq \int_X (\|a\|^\alpha + \|b\|^\alpha) \|a - b\|^\alpha \, d\mu(y).
\end{aligned}$$

So $|g(a) - g(b)| \leq \|a - b\|^\alpha (\|a\|^\alpha + \|b\|^\alpha)$ and g is continuous.

Now for the equicontinuity of $(g_m)_m$:

$$\begin{aligned}
|g_m(a) - g_m(b)| &= \|\text{ave} \left(g \left(\sum \theta_n a_n e_{\sigma(n)} \right) - g \left(\sum \theta_n b_n e_{\sigma(n)} \right) \right)\| \\
&\leq \text{ave} \|g \left(\sum \theta_n a_n e_{\sigma(n)} \right) - g \left(\sum \theta_n b_n e_{\sigma(n)} \right)\| \\
&\leq \text{ave} \left(\left\| \sum \theta_n a_n e_{\sigma(n)} - \sum \theta_n b_n e_{\sigma(n)} \right\|^\alpha \right. \\
&\quad \cdot \left. \left(\left\| \sum \theta_n a_n e_{\sigma(n)} \right\|^\alpha + \left\| \sum \theta_n b_n e_{\sigma(n)} \right\|^\alpha \right) \right) \\
&= \text{ave} (\|a - b\|^\alpha (\|a\|^\alpha + \|b\|^\alpha)) \\
&= \|a - b\|^\alpha (\|a\|^\alpha + \|b\|^\alpha).
\end{aligned}$$

So by Ascoli's theorem, there is a subsequence $(g_{m_k})_k$ of $(g_m)_m$ which converges pointwise to a continuous function \tilde{g} . The property of the g_m 's implies that \tilde{g} must necessarily be symmetric. We have that $\tilde{g}(0) = 0$, and that $\phi_1(\|x\|) \leq \tilde{g}(x) \leq \phi_2(\|x\|)$. Finally, as it is easily checked that the g_m 's are negative definite functions, it also follows easily that \tilde{g} is a negative definite function.

Step 4.5.6 (Final step). Using the relationship between negative and positive definite kernels given by Schoenberg's Lemma (Lemma 4.2.5), we get that the function $\tilde{f} = e^{-\tilde{g}}$ is a symmetric continuous positive definite function on X .

Since we have

$$\liminf_{n \rightarrow \infty} \frac{\|e_1 + e_2 + \cdots + e_n\|}{\sqrt{n}} = 0,$$

we conclude by Theorem 4.4.2 of Aharoni, Maurey and Mityagin that \tilde{f} is constant.

On the other hand, $\tilde{f}(0) = e^{-\tilde{g}(0)} = 1$, while $0 \leq \tilde{f}(x) \leq e^{-\phi_1(\|x\|)} \rightarrow 0$ as $\|x\| \rightarrow \infty$.

This gives a contradiction and finishes the proof.

□

CHAPTER V

CHARACTERIZATION OF SPACES THAT COARSELY EMBED INTO A HILBERT SPACE

In this chapter, we continue the work of the previous chapter by giving a characterization of quasi-Banach spaces that coarsely embed into a Hilbert space.

Let $(\Omega, \mathcal{B}, \mu)$ be a probability space. We denote by $L_0(\mu)$ the space of all measurable functions endowed with the topology of convergence in measure.

5.1 Quasi-Banach spaces

Definition 5.1.1. A *quasi-norm* on a linear space X is a map $\|\cdot\| : X \rightarrow [0, \infty)$ that satisfies for every $x, y \in X$:

1. $\|x\| = 0$ if and only if $x = 0$,
2. $\|\lambda x\| = |\lambda|\|x\|$ for every scalar λ ,
3. $\|x + y\| \leq K(\|x\| + \|y\|)$, where the constant $K \geq 1$ does not depend on x, y .

A *quasi-Banach space* is a linear space X endowed with a quasi-norm under which it is complete.

Aoki [Ao] and Rolewicz [R] characterized quasi-Banach spaces as follows:

Theorem 5.1.2 (Aoki-Rolewicz Theorem). *Let X be a quasi-Banach space. Then there exists $0 < p \leq 1$ and an equivalent quasi-norm $\|\cdot\|$ on X that satisfies for every $x, y \in X$:*

$$\|x + y\|^p \leq \|x\|^p + \|y\|^p.$$

Idea of the proof: Write $|\cdot|$ the original quasi-norm on X , and denote by $k = \inf\{K \geq 1, \forall x, y \in X, |x + y| \leq K(|x| + |y|)\}$. It is shown (see e.g. [KPR]) that the function $\|\cdot\|$ defined on X by:

$$\|x\| = \inf \left\{ \left(\sum_{i=1}^n |x_i|^p \right)^{\frac{1}{p}}, x = \sum_{i=1}^n x_i \right\},$$

where p is such that $2^{\frac{1}{p}} = 2k$, is an equivalent quasi-norm on X that satisfies the required inequality. \square

Remark 5.1.3. It then follows that X under this new quasi-norm is of type p , meaning that for every $x_1, x_2, \dots, x_n \in X$,

$$\left(\text{ave} \left\| \sum_{i=1}^n \theta_i x_i \right\|^p \right)^{\frac{1}{p}} \leq \left(\sum_{i=1}^n \|x_i\|^p \right)^{\frac{1}{p}},$$

where the average is taken over all possible signs $\theta_i \in \{-1, +1\}$, $i \in \{1, 2, \dots, n\}$.

The next result we need is Nikisin's Theorem [Ni]:

Theorem 5.1.4 (Nikisin's Theorem). *Let X be a quasi-Banach space of type $0 < p \leq 1$ that is a subspace of $L_0(\mu)$ for some measure space $(\Omega, \mathcal{B}, \mu)$. Then X is isomorphic to a subspace of $L_r(\mu)$ for every $r < p$.*

5.2 Positive definite functions and the space $L_0(\mu)$

Here we relate the theory of positive definite functions with classical probability theory. In fact, we have the following two classical theorems: Bochner's Theorem (see e.g. [Ru, page 19]) and Kolmogorov's Consistency Theorem (see e.g. [GS, page 108]).

Theorem 5.2.1 (Bochner's Theorem). *A function $f : \mathbb{R}^n \rightarrow \mathbb{R}$ is positive definite if and only if it is the Fourier transform of a probability measure μ on \mathbb{R}^n , namely*

$$f(t) = \int_{\mathbb{R}^n} e^{i(t \cdot x)} d\mu(x).$$

Kolmogorov's Consistency Theorem proves the existence of measures on infinite product spaces.

Let X be any set, and let $\Omega = \mathbb{R}^X = \{\omega = (\omega_x)_{x \in X}, \omega_x \in \mathbb{R}\}$, and let \mathcal{B} be the product σ -algebra on Ω generated by the sets of the form $\{\omega = (\omega_x)_x : \omega_x \in (a_x, b_x] \text{ for } x \in A\}$, where $-\infty \leq a_x < b_x \leq \infty$, and A runs over all the finite subsets of X . Kolmogorov's Consistency Theorem states as follows:

Theorem 5.2.2 (Kolmogorov's Consistency Theorem). *Let μ_A be a probability measure on \mathbb{R}^A , where A runs over all the finite subsets of X . Assume that the probabilities $(\mu_A)_A$ satisfy the following consistency condition:*

$$(A \subset B) \Rightarrow (\forall E \subset \mathbb{R}^A, \mu_B(E \times \mathbb{R}^{B \setminus A}) = \mu_A(E)).$$

Then there is a unique probability measure μ on (Ω, \mathcal{B}) such that:

$$\forall A, \forall E \subset \mathbb{R}^A, \mu(E \times \mathbb{R}^{X \setminus A}) = \mu_A(E).$$

Bretagnolle, Dacunha and Krivine [BrDK] and Aharoni, Maurey and Mityagin [AMM] used those two classical theorems to get the following (see also [BeL, Proposition 8.7]):

Proposition 5.2.3. *Let X be a real linear metric space, and let $f : X \rightarrow \mathbb{R}$ be a continuous positive definite function that satisfies $f(0) = 1$. Then one can find a probability space $(\Omega, \mathcal{B}, \mu)$, and a continuous linear operator $U : X \rightarrow L_0(\mu)$ such that*

$$\forall t \in \mathbb{R}, \forall x \in X, f(tx) = \mathbb{E}(e^{itU(x)}).$$

Proof: For a finite subset A of X , write $\mathbb{R}^A = \{t = (t_a)_{a \in A}, t_a \in \mathbb{R}\}$, and define on \mathbb{R}^A a function f_A by setting $f_A(t) = \sum_{a \in A} t_a a$. Then f_A is a positive definite function

on \mathbb{R}^A and so by Bochner's theorem it is the Fourier transform of some probability measure μ_A on \mathbb{R}^A .

For $A \subset B$, make \mathbb{R}^A a subspace of \mathbb{R}^B in a natural way, and notice that the restriction $f_{B|_{\mathbb{R}^A}}$ of f_B to \mathbb{R}^A is equal to f_A . This gives that the probability measures $(\mu_A)_A$ form a consistent family as A runs through all the finite subsets of X . Applying Kolmogorov's Consistency Theorem, one can find a probability measure μ on \mathbb{R}^X and a family $(U_x)_{x \in X}$ of measurable functions such that for any finite subset A of X ,

$$f\left(\sum_{a \in A} t_a a\right) = \int_{\mathbb{R}^X} e^{i \sum_{a \in A} t_a U_a(\omega)} d\mu(\omega).$$

Let us check that the map $U : X \rightarrow L_0(\mu)$ which associates U_x to x is linear and continuous.

Let $x_1, x_2 \in X$ and $\lambda_1, \lambda_2 \in \mathbb{R}$. We have for every $t \in \mathbb{R}$:

$$\mathbb{E}\left(e^{it(U_{\lambda_1 x_1 + \lambda_2 x_2} - \lambda_1 U_{x_1} - \lambda_2 U_{x_2})}\right) = f(t(\lambda_1 x_1 + \lambda_2 x_2) - (t\lambda_1)x_1 - (t\lambda_2)x_2) = f(0) = 1,$$

so $U_{\lambda_1 x_1 + \lambda_2 x_2} = \lambda_1 U_{x_1} + \lambda_2 U_{x_2}$.

Assume $x_n \rightarrow 0$. Then $\mathbb{E}(itU_{x_n}) = f(tx_n) \rightarrow f(0) = 1$ by the continuity of f .

So $(U_{x_n})_n$ converges to 0 in measure. \square

5.3 Main result

Proposition 5.3.1. *Let X be a quasi-Banach space which coarsely embeds into a Hilbert space. Then there exists on X a continuous negative definite function g which satisfies $g(0) = 0$ and $\phi_1(\|x\|) \leq g(x) \leq \|x\|^{2\alpha}$ where $\phi_1 : [0, \infty) \rightarrow [0, \infty)$ is a nondecreasing function satisfying $\phi_1(t) \rightarrow \infty$ as $t \rightarrow \infty$, and $\alpha > 0$.*

Proof: Steps 4.5.2, 4.5.3, 4.5.4 and the piece of Step 4.5.5 for the continuity of g extends to the case when X is a quasi-Banach space. \square

Theorem 5.3.2. *A quasi-Banach space X coarsely embeds into a Hilbert space if and only if there is a probability space $(\Omega, \mathcal{B}, \mu)$ such that X is linearly isomorphic to a subspace of $L_0(\mu)$.*

Proof: Let X be a quasi-Banach space. The Aoki-Rolewicz Theorem (Theorem 5.1.2) gives an equivalent quasi-norm $\|\cdot\|$ on X which is also p -subadditive for some $0 < p \leq 1$, i.e. $\|x + y\|^p \leq \|x\|^p + \|y\|^p$ for all x, y in X . In particular X under this norm has type p .

Say X is linearly isomorphic to a subspace of $L_0(\mu)$ for some probability space $(\Omega, \mathcal{B}, \mu)$. Then since X has type p , Nikisin's Theorem (Theorem 5.1.4) asserts that X is isomorphic to a subspace of $L_r(\mu)$ for every $r < p$. Now since $r < 2$, Nowak's result in [No] implies that X coarsely embeds into a Hilbert space. In fact, Nowak notices that the negative definite function $x \mapsto \|x\|^r$ on $L_r(\mu)$ when $r < 2$ gives, via Schoenberg's Theorem, a map into a Hilbert space which controls large (and small) distances uniformly, hence giving a coarse embedding of L_r into a Hilbert space. Mendel and Naor in [MeN] actually give an explicit formula for a (uniform and) coarse embedding of L_r into L_q when $r < q$ by $T : L_r(\mathbb{R}) \rightarrow L_q(\mathbb{R} \times \mathbb{R})$:

$$T(f)(s, t) = \frac{1 - e^{itf(s)}}{|t|^{(r+1)/q}}.$$

Conversely, let X be a quasi-Banach space which coarsely embeds into a Hilbert space. Let g be the negative definite function on X given by Proposition 5.3.1, and let f be the continuous positive definite function given by $f = e^{-g}$. Use Proposition 5.2.3 to get a probability space $(\Omega, \mathcal{B}, \mu)$ and a continuous linear operator $U : X \rightarrow L_0(\mu)$ such that the characteristic function $\mathbb{E} \exp(itUx)$ of Ux is equal to $f(tx)$ for every $x \in X$ and $t \in \mathbb{R}$. We show that U is an isomorphism into.

Let $(x_n)_n$ be a sequence in X such that $U(x_n) \rightarrow 0$ in $L_0(\mu)$, i.e. in measure.

Then $f(tx_n) = \mathbb{E}(\exp(itUx_n)) \rightarrow 1$ for each fixed t in \mathbb{R} . If $(x_n)_n$ does not converge to 0, then by passing to a subsequence we can assume without loss of generality that $\|x_n\| \geq \epsilon$ for all n and for some $\epsilon > 0$. But since ϕ_1 is nondecreasing, we get for every $t > 0$:

$$e^{-\phi_1(t\|x_n\|)} \leq e^{-\phi_1(t\epsilon)}.$$

Since $\phi_1(s) \rightarrow \infty$ as $s \rightarrow \infty$, we can pick $t_0 > 0$ so that $e^{-\phi_1(t_0\epsilon)} < \frac{1}{2}$. For that t_0 , we have for every n :

$$f(t_0x_n) \leq e^{-\phi_1(t_0\epsilon)} < \frac{1}{2}.$$

This contradicts the fact that $f(t_0x_n) \rightarrow 1$. Thus $x_n \rightarrow 0$, and hence U is one-to-one and its inverse is continuous. \square

Corollary 5.3.3. *A quasi-Banach space coarsely embeds into a Hilbert space if and only if it uniformly embeds into a Hilbert space.*

Proof: Aharoni, Maurey and Mityagin [AMM] proved that a quasi-Banach space uniformly embeds into a Hilbert space if and only if it is isomorphic to a linear subspace of $L_0(\mu)$ for some probability space $(\Omega, \mathcal{B}, \mu)$. The idea of the proof presented here for the coarse case was actually mirrored after their original proof for the uniform case. It has been a surprise to notice that practically the same proof works for the coarse case when uniform embeddings give information only on small distances, while coarse embeddings give information only on large distances.

CHAPTER VI

CONCLUSION

The results we presented here add to the theory of nonlinear geometry of Banach spaces that has been studied for many years, and also to the new trend that links it to geometric group theory. On the other hand, they leave some questions that need further investigations.

In regard to Chapter II, can ℓ_2 be a Lipschitz quotient of ℓ_p when $p < 2$? A negative answer to this would give that a Lipschitz quotient of ℓ_p has to be a linear quotient of ℓ_p when $p < 2$. This would follow the same line as the argument we presented in Chapter II.

Another question that is suggested by the present study is whether or not a Banach space that is uniformly homeomorphic to a subspace of ℓ_p has to be a subspace of ℓ_p when $p < 2$. This should follow from a positive answer to a more general question: is the modulus of asymptotic uniform convexity $\bar{\delta}_X$ of a Banach space X preserved under uniform homeomorphism? In fact, we can use an ultraproduct argument as in [HM] to see that the space in question is a subspace of L_p . And a subspace of L_p ($p < 2$) whose modulus of asymptotic uniform convexity is of power type p will be a subspace of ℓ_p by following a similar argument as in [GoKL2] for the proof of the uniform characterization of quotients of ℓ_p , $p > 2$. However, even the preservation of the modulus of asymptotic uniform convexity under a Lipschitz isomorphism is still unknown.

The question whether c_0 is minimal as a Banach space that contains a Lipschitz copy of every separable metric space is still wide open. In other words, if c_0 Lipschitz embeds into a Banach space X , must it linearly embed? Godefroy, Kalton and Lancien

gave a complete answer to this when the Lipschitz embedding is actually a Lipschitz isomorphism [GoKL1]. In such a case, X is actually isomorphic to c_0 . They also showed in [GoKL2] that if a uniform homeomorphism $f : c_0 \rightarrow X$ only performs a small enough deformation of c_0 in the sense of uniform distance, then X has to be isomorphic to c_0 . However, without this assumption the general question still remains open.

The coarse geometry of a Banach space and its relation to geometric group theory and algebraic topology has been the object of Chapters IV and V. Chapter V puts a close on the linear geometry of a quasi-Banach space that admits a coarse embedding into a Hilbert space. However, since the main link between geometric group theory and Banach space theory expressed here has its focus on metric spaces, it leaves this question open for general metric spaces. For example: does there exist a discrete metric space with bounded geometry that coarsely embeds into ℓ_p for $p > 2$ but not into ℓ_2 ? More generally, what characteristics must all metric subsets of a given Banach space have? Which of those characteristics are transported by coarse embeddings?

REFERENCES

- [A] I. Aharoni, Every separable metric space is Lipschitz equivalent to a subset of c_0 , Israel J. Math. 19 (1974), 284–291.
- [AMM] I. Aharoni, B. Maurey, B. S. Mityagin, Uniform embeddings of metric spaces and of Banach spaces into Hilbert spaces, Israel J. Math. 52:3 (1985), 251–265.
- [Ao] T. Aoki, Locally bounded linear topological spaces, Proc. Imp. Acad. Tokyo 18 (1942), 588–594.
- [BJLPS] S. M. Bates, W. B. Johnson, J. Lindenstrauss, D. Preiss, G. Schechtman, Affine approximation of Lipschitz functions and nonlinear quotients, Geom. Funct. Anal. 9:6 (1999), 1092–1127.
- [BeL] Y. Benyamini, J. Lindenstrauss, Geometric Nonlinear Functional Analysis, Coll. Pub. 48, Amer. Math. Soc., Providence, 2000.
- [BrDK] J. Bretagnolle, D. Dacunha Castelle, J. L. Krivine, Lois stables et espaces L^p , Ann. Inst. Henri Poincaré 2 (1966), 231–259.
- [GS] I. I. Gikhman, A. V. Skorokhod, Introduction to the Theory of Random Processes, W. B. Saunders Company, Philadelphia, 1969.
- [GoKL1] G. Godefroy, N. Kalton, G. Lancien, Subspaces of $c_0(\mathbb{N})$ and Lipschitz isomorphisms, Geom. Funct. Anal. 10:4 (2000), 798–820.
- [GoKL2] G. Godefroy, N. J. Kalton, G. Lancien, Szlenk Indices and Uniform Homeomorphisms, Trans. Amer. Math. Soc. 353:10 (2001), 3895–3918.
- [Gr1] M. Gromov, Asymptotic Invariants of Infinite Groups. Geometric Group Theory, Vol. 2, London Math. Soc. Lecture Note Series 182, Cambridge University Press, Cambridge, 1993.
- [Gr2] M. Gromov, Metric Structures for Riemannian and Non-Riemannian Spaces, Progress in Math. 152, Birkhauser, Boston, 1998.
- [H] S. Heinrich, Ultraproducts in Banach space theory, J. Reine Angew. Math. 313 (1980), 72–104.
- [HM] S. Heinrich, P. Mankiewicz, Applications of ultrapowers to uniform and Lipschitz classification of Banach spaces, Studia Math. 73 (1982), 225–251.
- [J] I. M. James, Introduction to uniform spaces, London Math. Soc. Lecture Note Series 144, Cambridge University Press, Cambridge, 1990.

- [JoLPS] W. B. Johnson, J. Lindenstrauss, D. Preiss, G. Schechtman, Almost Fréchet differentiability of Lipschitz mappings between infinite-dimensional Banach spaces, *Proc. London Math. Soc.* 84:3 (2002), 711–746.
- [JoLS] W. B. Johnson, J. Lindenstrauss, G. Schechtman, Banach spaces determined by their uniform structure, *Geom. Funct. Anal.* 6 (1996), 430–470.
- [JoO] W. B. Johnson, E. Odell, Subspaces of L_p which embed into ℓ_p , *Compositio Math.* 28 (1974), 37–49.
- [JoZ] W. B. Johnson, M. Zippin, On subspaces of quotients of $(\sum G_n)_{l_p}$ and $(\sum G_n)_{c_0}$. *Proceedings of the International Symposium on Partial Differential Equations and the Geometry of Normed Linear Spaces* (Jerusalem, 1972). *Israel J. Math.* 13 (1972), 311–316.
- [KPR] N. J. Kalton, N. T. Peck, J. W. Roberts, *An F -Space Sampler*, London Math. Soc. Lecture Notes 89, Cambridge University Press, Cambridge, 1984.
- [KaY] G. Kasparov, G. Yu, The coarse geometric Novikov conjecture and uniform convexity, preprint (2004), email communication, Vanderbilt University, Nashville.
- [M] B. Maurey, Un théorème de prolongement, *C. R. Acad. Sci. Paris Sér. A* 279 (1974), 329–332.
- [MaU] S. Mazur, S. Ulam, Sur les transformations isométriques d’espaces vectoriels normés, *C. R. Acad. Sci. Paris* 194 (1932), 946–948.
- [MeN] M. Mendel, A. Naor, Euclidian quotients of finite metric spaces, *Adv. Math.* 189 (2004), 451–494.
- [Mi] V. Milman, Geometric theory of Banach spaces. II. Geometry of the unit ball, (Russian) *Uspehi Mat. Nauk* 26:6 (1971), 73–149. English Translation: *Russian Math. Surveys* 26:6 (1971), 79–163.
- [NPSS] A. Naor, Y. Peres, O. Schramm, S. Sheffield, Markov chains in smooth Banach spaces and Gromov hyperbolic metric spaces, preprint (2004), arXiv:math.FA/0410422.
- [Ni] E. M. Nikišin, Resonance theorems and expansions in eigenfunctions of the Laplace operator (Russian), *Izv. Akad. Nauk SSSR Ser. Mat.* 36 (1972), 795–813. English translation: *Math. USSR-Izv.* 6 (1972), 788–806.
- [No] P. Nowak, Coarse embeddings of metric spaces into Banach spaces, preprint (2004), arXiv:math.MG/0404401.
- [R] S. Rolewicz, On a certain class of linear metric spaces, *Bull. Acad. Polon. Sci. Sér. Sci. Math. Astrono. Phys.* 5 (1957), 471–473.

- [Ru] W. Rudin, Fourier Analysis on Groups, Interscience Publishers, New York, 1962.
- [S] I. J. Schoenberg, Metric spaces and positive definite functions, Trans. Amer. Math. Soc. 44 (1938), 522–536.
- [WW] J. H. Wells, L. R. Williams, Embeddings and Extensions in Analysis, Ergebnisse der Mathematik und ihrer Grenzgebiete 84, Springer-Verlag, New York, 1975.
- [Y] G. Yu, The coarse Baum-Connes conjecture for spaces which admit a uniform embedding into Hilbert space, Invent. Math. 139:1 (2000), 201–240.

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